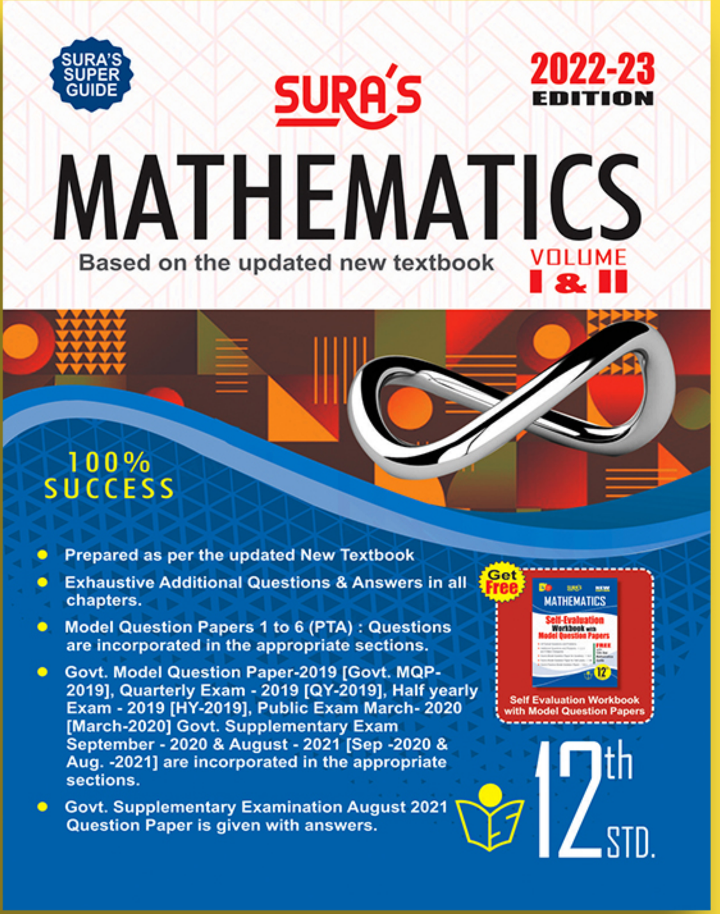

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Authors :

Mr. G.Selvaraj, M.Sc., M.Ed., M.Phil.,
Chennai

Edited by :

Mr. S. Sathish M.Sc., M.Phil.,

Reviewed by :

Mr.S. Niranjan, B.Tech., (NITT)PGDM (IIM)
Chennai

Head Office:

1620, 'J' Block, 16th Main Road,
Anna Nagar, **Chennai - 600 040.**
Phones: 044-4862 9977, 044-486 27755.
Mob : 81242 01000/ 81243 01000
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PREFACE

Sir,
An equation has no meaning, for me
unless it expresses a thought of GOD - Ramanujam
[Statement to a friend]

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers,

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **Sura's Mathematics Guide Volume I and Volume II** for +2 Standard – Edition 2022-23. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises and in-text questions in addition to precise answers for textual questions.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

Subash Raj, B.E., M.S.
- Publisher
Sura Publications

All the Best

CONTENTS

CHAPTER
1

APPLICATIONS OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A.
- If $AB = BA = I_n$, then the matrix B is called the inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon** form if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [$\rho(A)$].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations.
- A system of linear equations having atleast one solution is said to be **consistent**.
- A system of linear equations having no solutions is said to be **inconsistent**.

IMPORTANT FORMULAE TO REMEMBER

- Co – factor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}
- For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$
  $AA^{-1} = A^{-1}A = I_n$
- If A is non – Singular then
  (i) $|A^{-1}| = \dfrac{1}{|A|}$
  (ii) $(A^T)^{-1} = (A^{-1})^T$
  (iii) $(\lambda A^{-1}) = \dfrac{1}{\lambda} A^{-1}$ where λ is a non – zero scalar.

Reversal law for inverses :

- $(AB)^{-1} = B^{-1}A^{-1}$ where A, B are non – singular matrices of same order.

Law of double inverse :

- If A is non - singular, A^{-1} is also non – singular and $(A^{-1})^{-1} = A$.
- If A is a non - singular square matrix of order n, then
  (i) $(\text{adj } A)^{-1} = \text{adj } (A^{-1}) = \dfrac{1}{|A|} \cdot A$
  (ii) $|\text{adj } A| = |A|^{n-1}$
  (iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$
  (iv) $\text{adj } (\lambda A) = \lambda^{n-1} \text{adj } (A)$ where λ is a non – zero scalar
  (v) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$
  (vi) $(\text{adj } A)^T = \text{adj } (A^T)$
- If a matrix contains at least one non – zero element, then $\rho(a) \geq 1$.
- The rank of identity matrix I_n is n.
  If A is an $m \times n$ matrix then $\rho(A) \leq \min\{m, n\}$.
- A square matrix A of order n is invertible if and only if $\rho(A) = n$.
- Transforming a non-singular matrix A to the form I_n, by applying row operations is called Gauss – Jordan method.

Matrix – Inversion method :

- The solution for $AX = B$ is $X = A^{-1}B$ where A and B are square matrices of same order and non – singular

Cramer's Rule :

- If $\Delta = 0$, Cramer's rule cannot be applied $x_1 = \dfrac{\Delta_1}{\Delta}, x_2 = \dfrac{\Delta_2}{\Delta}, x_3 = \dfrac{\Delta_3}{\Delta}$

Gaussian Elimination method :

Transform the augmented matrix of the system of linear equations into row – echelon form and then solve by back substitution method.

Rouches capelli Theorem :

A system of equations $AX = B$ is consistent if and if $\rho(A) = \rho([A|B])$

(i) If $\rho(A) = \rho([A|B]) = n$, the number of unknowns, then the system is consistent and has a unique solution.

(ii) If $\rho(A) = \rho([A|B]) = n - k$, $k \neq 0$ then the system is consistent and has infinitely many solutions.

(iii) If $\rho(A) \neq \rho([A|B])$, then the system is inconsistent and has no solution.

Homogeneous system of linear equations :

(i) If $\rho(A) = \rho([A|B]) = n$, then the system has a unique solution which is the trivial solution for trivial solution, $|A| \neq 0$

(ii) If $\rho(A) = \rho([A|B]) < n$, the system has a non – trivial solution.

For non – trivial solution, $|A| = 0$.

- $A^{-1} = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj } A$
- $A = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj (adj A)}$

EXERCISE 1.1

1. Find the adjoint of the following :

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ **(iii)** $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Sol. **(i)** Let $A = \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

$\text{adj } A = \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$

[Interchange the elements in the leading diagonal and change the sign of the elements in off diagonal]

(ii) Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $\lambda = \dfrac{1}{3}$

Since adj $(\lambda A) = \lambda^{n-1}$(adj A)

we get adj $\left(\dfrac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\right) = \left(\dfrac{1}{3}\right)^2$

$$\text{adj}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

∴ Required adjoint matrix

$$= \frac{1}{9}\begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} (2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & -(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{3}{9}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

[Taking 3 common from each entry]

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. **Find the inverse (if it exists) of the following :**

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Sol. **(i)** Let A $= \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Since A is non – singular, A^{-1} exists

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{Now, adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

[Inter change the entries in leading diagonal and change the sign of elements in the off diagonal]

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) Let A $= \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Expanding along R_1,

$$|A| = 5\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 5\,(25 - 1) - 1\,(5 - 1) + 1\,(1 - 5)$$

$$= 5\,(24) - 1\,(4) + 1\,(-4)$$

$$= 120 - 4 - 4 = 120 - 8 = 112 \neq 0$$

Since A is non singular, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(25-1)-(5-1)+(1-5) \\ -(5-1)+(25-1)-(5-1) \\ +(1-5)-(5-1)+(25-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

Taking 4 common from every entry we get,

$$\text{adj } A = 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{112} \cdot 4 \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$= \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) Let A $= \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Expanding along R_1 we get,

$$|A| = 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$$= 2\,(8 - 7) - 3\,(6 - 3) + 1\,(21 - 12)$$

$$= 2\,(1) - 3(3) + 1\,(9)$$

$$= 2 - \cancel{9} + \cancel{9} = 2 \neq 0$$

Since A is a non-singular matrix, A^{-1} exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(8-7)-(6-3)+(21-12) \\ -(6-7)+(4-3)-(14-9) \\ +(3-4)-(2-3)+(8-9) \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. **If F (α) = $\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$** **[Hy - 2019]**

Sol. Given that F (α) = $\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

Expanding along R_1 we get,

$$|F(\alpha)| = \cos\alpha \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 + \sin\alpha \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$$

$= \cos\alpha\,(\cos - 0) + \sin\alpha\,(0 + \sin\alpha)$

$= \cos^2 + \sin^2\alpha = 1 \neq 0$

Since F (α) is a non-singular matrix, $[F(\alpha)]^{-1}$ exists.

Now, adj (F(α)) =

$$\begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ \sin\alpha & \cos\alpha \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{vmatrix} & +\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} & -\begin{vmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & \sin\alpha \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(\cos\alpha - 0) & -(0) & +(0 + \sin\alpha) \\ -(0) & +(\cos^2\alpha + \sin^2\alpha) & -(0) \\ +(0 - \sin\alpha) & -(0) & +(\cos - 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & 0 & +\sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\therefore F(\alpha)^{-1} = \frac{1}{|F(\alpha)|} \text{ adj } (F(\alpha))$$

$$[F(\alpha)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad ...(1)$$

Now, F(– α) = $\begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad ... (2)$$

[∵ cosα is an even function, cos (–α) = cos α and sin α is an odd function, sin (–α) = –sinα]

From (1) and (2)

$[F(\alpha)]^{-1} = F(-\alpha)$

Hence proved.

4. **If A=$\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.**

Sol. Given A = $\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$\therefore A^2 - 3A - 7I_2$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9+0 \\ -3+3+0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$$

Hence proved.

$\therefore A^2 - 3A - 7I_2 = 0$

Post – multiplying by A^{-1} we get,

$A^2 . A^{-1} - 3AA^{-1} - 7I_2\, A^{-1} = 0.A^{-1}$

$\Rightarrow A(AA^{-1}) - 3\,(AA^{-1}) - 7(A^{-1}) = 0$

[∵ $I_2\, A^{-1} = A^{-1}$ and $(0)A^{-1} = 0$]

$\Rightarrow AI - 3I - 7A^{-1} = 0$ [∵ $AA^{-1} = I$]

$\Rightarrow AI - 3I = 7A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{7}[A - 3I]$ [∴ AI = A]

$$\Rightarrow A^{-1} = \frac{1}{7}\left[\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. **If $A = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1} = A^{T}$.**

Sol.

$$\text{To prove } A^{-1} = A^{T}$$

$$AA^{-1} = AA^{T}$$

It is enough to prove $AA^{T} = I$

$$AA^{T} = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix}64+1+16 & -32+4+28 & -8-8+16\\ -32+4+28 & 16+16+49 & 4-32-28\\ -8-8+16 & 4-32+28 & 1+64+16\end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix}81 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 81\end{bmatrix} = \frac{81}{81}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

Therefore $A^{-1} = A^{T}$

6. **If $A = \begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_2$.** [Sep. - 2020; Aug. - 2021]

Sol.

$$\text{Given } A = \begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix}3 & 4\\ 5 & 8\end{bmatrix}$$

[Interchange the elements in the leading diagonal and change the sign of the elements in the off diagonal]

$$|A| = 24 - 20 = 4$$

$$\therefore A\,(\text{adj } A) = \begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}\begin{bmatrix}3 & 4\\ 5 & 8\end{bmatrix}$$

$$= \begin{bmatrix}24-20 & 32-32\\ -15+15 & -20+24\end{bmatrix} = \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} \quad ...(1)$$

$$(\text{adj } A)\,(A) = \begin{bmatrix}3 & 4\\ 5 & 8\end{bmatrix}\begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}$$

$$= \begin{bmatrix}24-20 & -12+12\\ 40-40 & -20+24\end{bmatrix} = \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} \quad ...(2)$$

$$|A|\, I_2 = 4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} \quad ...(3)$$

From (1), (2) and (3), it is proved that $A\,(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_2$ is verified.

7. **If $A = \begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$ and $B = \begin{bmatrix}-1 & -3\\ 5 & 2\end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.**

Sol. Given $A = \begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$ and $B = \begin{bmatrix}-1 & -3\\ 5 & 2\end{bmatrix}$

$$AB = \begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}\begin{bmatrix}-1 & -3\\ 5 & 2\end{bmatrix} = \begin{bmatrix}-3+10 & -9+4\\ -7+25 & -21+10\end{bmatrix}$$

$$= \begin{bmatrix}7 & -5\\ 18 & -11\end{bmatrix}$$

$$|AB| = -77 + 90 = 13 \neq 0 \Rightarrow (AB)^{-1} \text{ exists}$$

$$|A| = 15 - 14 = 1 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$|B| = -2 + 15 = 13 \neq 0 \Rightarrow B^{-1} \text{ exists}$$

$$(AB)^{-1} = \frac{1}{|AB|}\,\text{adj }(AB) = \frac{1}{13}\begin{pmatrix}-11 & 5\\ -18 & 7\end{pmatrix} \quad ...(1)$$

$$B^{-1} = \frac{1}{|B|}(\text{adj } B)$$

$$= \frac{1}{13}\begin{pmatrix}2 & 3\\ -5 & -1\end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{1}\begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix}$$

$$= \begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix}$$

$$\therefore B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix}2 & 3\\ -5 & -1\end{pmatrix}\begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix}10-21 & -4+9\\ -25+7 & 10-3\end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix}-11 & 5\\ -18 & 7\end{pmatrix} \quad ...(2)$$

From (1) and (2) it is prove that $(AB)^{-1} = B^{-1}A^{-1}$.

8. **If $\text{adj }(A) = \begin{bmatrix}2 & -4 & 2\\ -3 & 12 & -7\\ -2 & 0 & 2\end{bmatrix}$, find A.**

Sol. Given $\text{adj } A = \begin{bmatrix}2 & -4 & 2\\ -3 & 12 & -7\\ -2 & 0 & 2\end{bmatrix}$

We know that $A = \pm\frac{1}{\sqrt{|\text{adj}\,A|}}.\text{adj (adj A)}$ ...(1)

$$|\text{adj A}| = 2\begin{vmatrix}12 & -7\\0 & 2\end{vmatrix} + 4\begin{vmatrix}-3 & -7\\-2 & 2\end{vmatrix} + 2\begin{vmatrix}-3 & 12\\-2 & 0\end{vmatrix}$$

[Expanded along R_1]

$$= 2\,(24-0) + 4\,(-6-14) + 2\,(0+24)$$

$$= 2\,(24) + 4\,(-20) + 2\,(24) = 48 - 80 + 48$$

$$= 96 - 80 = 16$$

Now, adj (adj A)

$$= \begin{bmatrix} +\begin{vmatrix}12 & -7\\0 & 2\end{vmatrix} - \begin{vmatrix}-3 & -7\\-2 & 2\end{vmatrix} + \begin{vmatrix}-3 & 12\\-2 & 0\end{vmatrix} \\ -\begin{vmatrix}-4 & 2\\0 & 2\end{vmatrix} + \begin{vmatrix}2 & 2\\-2 & 2\end{vmatrix} - \begin{vmatrix}2 & -4\\-2 & 0\end{vmatrix} \\ +\begin{vmatrix}-4 & 2\\12 & -7\end{vmatrix} - \begin{vmatrix}2 & 2\\-3 & -7\end{vmatrix} + \begin{vmatrix}2 & -4\\-3 & 12\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix} +(24-0)-(-6-14)+(0+24) \\ -(-8-0)+(4+4)-(0-8) \\ +(28-24)-(-14+6)+(24-12)\end{bmatrix}^T$$

$$= \begin{bmatrix}24 & 20 & 24\\8 & 8 & 8\\4 & 8 & 12\end{bmatrix}^T$$

$$= \begin{bmatrix}24 & 8 & 4\\20 & 8 & 8\\24 & 8 & 12\end{bmatrix}$$

$$= 4\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix} \quad ...(3)$$

Substituting (2) and (3) in (1) we get,

$$A = \pm\frac{1}{\sqrt{16}}.4\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix}$$

$$A = \pm\frac{4}{4}\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix}$$

$$= \pm\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix}$$

9. If adj(A) = $\begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$, find A^{-1}. **[PTA-6]**

Sol. Given adj (A) $= \begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$

We know that $A^{-1} = \pm\frac{1}{\sqrt{|\text{adj}A|}}(\text{adj A})$ ...(1)

$$|\text{adj A}| = 0 + 2\begin{vmatrix}6 & -6\\-3 & 6\end{vmatrix} + 0$$

[Expanded along R_1]

$$= 2\,(36-18) = 2(18) = 36$$

$$\therefore A^{-1} = \pm\frac{1}{\sqrt{36}}\begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$$

$$= \pm\frac{1}{6}\begin{bmatrix}0 & -2 & 0\\6 & 2 & -6\\-3 & 0 & 6\end{bmatrix}$$

10. Find adj (adj (A)) if adj A = $\begin{bmatrix}1 & 0 & 1\\0 & 2 & 0\\-1 & 0 & 1\end{bmatrix}$.

Sol. Given adj A = $\begin{bmatrix}1 & 0 & 1\\0 & 2 & 0\\-1 & 0 & 1\end{bmatrix}$

$$\text{Now adj(adj A)} = \begin{bmatrix} +\begin{vmatrix}2 & 0\\0 & 1\end{vmatrix} - \begin{vmatrix}0 & 0\\-1 & 1\end{vmatrix} + \begin{vmatrix}0 & 2\\-1 & 0\end{vmatrix} \\ -\begin{vmatrix}0 & 1\\0 & 1\end{vmatrix} + \begin{vmatrix}1 & 1\\-1 & 1\end{vmatrix} - \begin{vmatrix}1 & 0\\-1 & 0\end{vmatrix} \\ +\begin{vmatrix}0 & 1\\2 & 0\end{vmatrix} - \begin{vmatrix}1 & 1\\0 & 0\end{vmatrix} + \begin{vmatrix}1 & 0\\0 & 2\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix}+(2-0) & -(0) & +(0+2)\\-(0) & +(1+1) & -(0)\\+(0-2) & -(0) & +(2-0)\end{bmatrix}^T = \begin{bmatrix}2 & 0 & 2\\0 & 2 & 0\\-2 & 0 & 2\end{bmatrix}^T$$

$$\text{adj (adj A)} = \begin{bmatrix}2 & 0 & -2\\0 & 2 & 0\\2 & 0 & 2\end{bmatrix}$$

CHAPTER

2

COMPLEX NUMBERS

MUST KNOW DEFINITIONS

- If a Complex number is of the form $x + iy$ where x is a real part and y is the imaginary part of the complex number.
- $z_1 = z_2$ if Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2)

Properties of complex numbers:

Under Additions

- Let z_1, z_2 and z_3 are complex numbers.
  - (i) Closure property ($z_1 + z_2$ is a complex number)
  - (ii) Commutative property ($z_1 + z_2 = z_2 + z_1$)
  - (iii) Associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
  - (iv) Additive identity $(z + 0 = 0 + z = z)$
  - (v) Additive inverse $(z + - z) = (- z + z) = 0$
- **Under Multiplication:**
  - (i) Closure property ($z_1\ z_2$ is also a complex number)
  - (ii) Commutative property $(z_1\ z_2 = z_2\ z_1)$
  - (iii) Associative property $(z_1\ z_2)\ z_3 = z_1\ (z_2\ z_3)$
  - (iv) Multiplicative identity $(z_1 \,.\, 1 = 1.\ z_1 = z_1)$
  - (v) Multiplicative inverse $z.w = w.z = 1 \Rightarrow w = z^{-1}$
- **Distributive property (Multiplication distributes over addition)**

  $z_1\ (z_2 + z_3) = z_1\ z_2 + z_1\ z_3$

  Also, $(z_1 + z_2)\ z_3 = z_1\ z_3 + z_2\ z_3$
- Conjugate of $x + iy$ is $x - iy$
- If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$
- $|z - z_0| = r$ is the equation of circle where z_0 is a fixed complex number and r is the distance from z_0 to z.
- Polar form of $z = x + iy$ is $z = r\,(\cos\theta + i \sin\theta)$

  where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.

De Moivre's theorems

- Given any complex number cos θ + i sin θ and any integer n, then (cos θ + i sin θ)^n = cos nθ + i sin nθ.
- $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$, $k = 0, 1, 2,\ldots\ldots n-1$.

IMPORTANT FORMULA TO REMEMBER

- $i^0 = 1, i^1 = i, i^4 = 1, i^{-4} = 1, (i)^{-1} = -i, (i)^{-2} = -1, (i)^{-3} = i$
- $\sqrt{ab} = \sqrt{a}.\sqrt{b}$ is valid only if atleast one of a, b is non-negative.
- When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$
$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$$
$$z_1\,z_2 = (x_1\,x_2 - y_1\,y_2) + i\,(x_1\,y_2 + x_2\,y_1)$$

Properties of complex conjugates

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(3) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(4) $\left(\frac{\overline{z_1}}{\overline{z_2}}\right) = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

(5) Re $(z) = \frac{z+\overline{z}}{2}$

(6) Im $(z) = \frac{z-\overline{z}}{2i}$

(7) $\left(\overline{z^n}\right) = \left(\overline{z}\right)^n$,where n is an integer.

(8) z is real iff $z = \overline{z}$

(9) z is purely imaginary iff $z = -\overline{z}$

(10) $\overline{\overline{z}} = z$

Properties of modulus of a complex number

(1) $|z| = |\overline{z}|$

(2) $|z_1 + z_2| \leq |z_1| + |z_2|$
(Triangle inequality)

(3) $|z_1 z_2| = |z_1|\,|z_2|$

(4) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(6) $|z^n| = |z|^n$, where n is an integer

(7) Re (z) $\leq |z|$

(8) Im (z) $\leq |z|$

- $|z_1 - z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
- $|z_1 - z_2| \leq |z_1| + |z_2|$
- $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- $\big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
- $|z_1 z_2| = |z_1|\,|z_2|$

- Let $a + ib = \sqrt{x+iy}$ then

  $x = \pm\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}}$,

  $y = \pm\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}$
- $|z - z_0| < r$ represents the points interior of the circle.
- $|z - z_0| > r$ represents the points exterior of the circle.
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- $\arg(z^n) = n \arg z$
- Alternate form of $\cos\theta + i\sin\theta$ is $\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)$, $k \in \mathbb{Z}$

Euler's formula

- $e^{i\theta} = \cos\theta + i\sin\theta$ or $z = re^{i\theta}$
- If $z = r(\cos\theta + i\sin\theta)$ then

  $z^{-1} = \dfrac{1}{r}(\cos\theta - i\sin\theta)$
- $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$
- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ and $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$
- $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$ and $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$
- $1 + \omega + \omega^2 + \ldots\ldots + \omega^{n-1} = 0$

  where ω is the n^{th}--- root of unity.

  $1\omega.\omega^2 \ldots.\omega^{n-1} = (-1)^{n-1}$

  $\omega^{n-k} = \omega^{-k} = (\bar{\omega})^k \quad 0 \le k \le n-1$

EXERCISE 2.1

Simplify the following:

1. $i^{1947} + i^{1950}$ **2.** $i^{1948} - i^{-1869}$

3. $\sum_{n=1}^{12} i^n$ **4.** $i^{59} + \dfrac{1}{i^{59}}$

5. $i\, i^2\, i^3 \ldots.\, i^{2000}$ **6.** $\sum_{n=1}^{10} i^{n+50}$

Sol. **1.** $i^{1947} + i^{1950} = i^{1944}.\, i^3 + i^{1948}.\, i^2$

[∵ 1944 is a multiple of 4 and1948 is also a multiple of 4]

$= (i^4)^{486}\,.\, i^2\,.\, i^1 + (i^4)^{487}\,.\, i^2 \; [i^4 = 1]$

$= (1^{486})\,(-1)\,(i) + (1)^{487}\,(-1) \; [i^2 = -1]$

$= -i - 1$

$= -1 - i$

2. $i^{1948} - i^{-1869} = (i^4)^{487} - [i^{-1868}\,.\, i^{-1}]$

$= 1^{487} - \left[\left(i^4\right)^{-467} . \dfrac{1}{i}\right] \; \left[\because i^4 = 1\right]$

$= 1 - \left[1.(-i)\right] \qquad \left[\because i^{-1} = \dfrac{1}{i} = -i\right]$

[One power any number is 1]

$= 1 + i$

3. $\sum_{n=1}^{12} i^n = (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12})$

$= (i - 1 - i + 1) + (i^{4+1} + i^{4+2} + i^{4+3} + (i^4)^2) + (i^{8+1} + i^{8+2} + i^{8+3} + (i^4)^3)$

$= 0 + (i + i^2 + i^3 + i^4) + (i^1 + i^2 + i^3 + i^4)$ $[\therefore i^2 = -1, i^3 = -i, i^4 = 1]$

$= 0 + (i - 1 - i + 1) + (i - 1 - i + 1)$

$= 0 + 0 + 0 = 0$

4. $i^{4\times14+3} + i^{-(4\times14+3)}$ **[Hy - 2019]**

$= (i^4)^{14} . i^3 + (i^4)^{-14} . i^{-3}$

$= 1. i^3 + 1.i^{-3}$ $[\because i^4 = 1]$

$= -i + i$ $[\because i^3 = -i \text{ and } i^{-3} = i]$

$= 0.$

5. $i. i^2\, i^3 \ldots\ldots i^{2000} = i^{1+2+3+\ldots\ldots+2000}$

$= i^{\frac{2000\times2001}{2}}$ $[\because 1 + 2 + 3 + \ldots n = \frac{n(n+1)}{2}]$

$= i^{1000\times2001} = i^{2001000} = 1$

$[\because$ 2001000 is divisible by 4 as its last two digits are divisible by 4]

6. $i^{1+50} + i^{2+50} + \ldots\ldots. + i^{10+50}$

$= i^{51} + i^{52} + \ldots\ldots. + i^{60}$

Taking i^{50} common we get,

$i^{50} [(i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + i^9 + i^{10}]$

$= i^{50} [0 + (i^{4+1} + i^{4+2} + i^{4+3} + i^{4+4}) + (i^{8+1} + i^{8+2})]$

$= i^{50} [0 + 0 + i + i^2]$ $[\because i + i^2 + i^3 + i^4 = 0]$

$= i^{50} [i - 1] = i^{48+2} (i - 1)$

$= i^2 (i - 1)$ $[\because i^{48} = 1]$

$= -1 (i - 1) = -i + 1 = 1 - i$

EXERCISE 2.2

1. Evaluate the following if $z = 5 - 2i$ and $w = -1 + 3i$

(i)	$z + w$	**(ii)**	$z - iw$
(iii)	$2z + 3w$	**(iv)**	$z\,w$
(v)	$z^2 + 2zw + w^2$	**(vi)**	$(z + w)^2$

Sol. **(i)** $z + w$

$= (5 - 2i) + (-1 + 3i) = (5 - 1) + i(-2 + 3)$

$= 4 + i(1) = 4 + i$

(ii) $z - iw$

$= (5 - 2i) - i(-1 + 3i)$

$= (5 - 2i) + (+i - 3i^2)$

$= 5 - 2i + i - 3(-1) = 5 - i + 3 = 8 - i$

(iii) $2z + 3w$

$= 2(5 - 2i) + 3(-1 + 3i) = 10 - 4i - 3 + 9i$

$= (10 - 3) + i(-4 + 9) = 7 + 5i$

(iv) $z\,w$

$= (5 - 2i)(-1 + 3i) = -5 + 15i + 2i - 6i^2$

$= -5 + 17i - 6(-1)$

$= -5 + 17i + 6 = 1 + 17i$

(v) $z^2 + 2zw + w^2$

$= (5 - 2i)^2 + 2(5 - 2i)(-1 + 3i) + (-1 + 3i)^2$

$= 25 + 4i^2 - 20i + 2[-5 + 15i + 2i - 6i^2] + 1 + 9i^2 - 6i$

$= 25 - 4 - 20i + 2(-5 + 17i + 6) + 1 - 9 - 6i$ $[\because i^2 = -1]$

$= 21 - 20i + 2(1 + 17i) - 8 - 6i$

$= 21 - 20i + 2 + 34i - 8 - 6i = 15 + 8i$

(vi) $(z + w)^2$

$= [(5 - 2i) + (-1 + 3i)]^2 = (4 + i)^2$

$= 16 + 8i - 1 = 16 - 1 + 8i = 15 + 8i$

2. Given the complex number $z = 2 + 3i$, represent the complex numbers in Argand diagram.

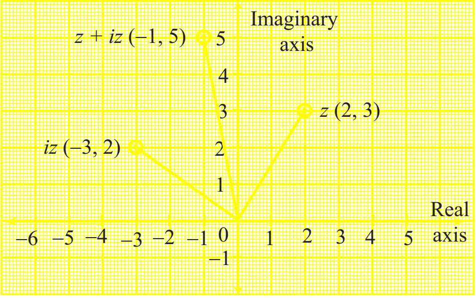
(i) z, iz, and $z + iz$ **(ii) z, $-iz$, and $z - iz$.**

Sol. **(i)** Represent z, iz and $z + iz$ in the Argand diagram.

$z = 2 + 3i$ can be represented as (2, 3)

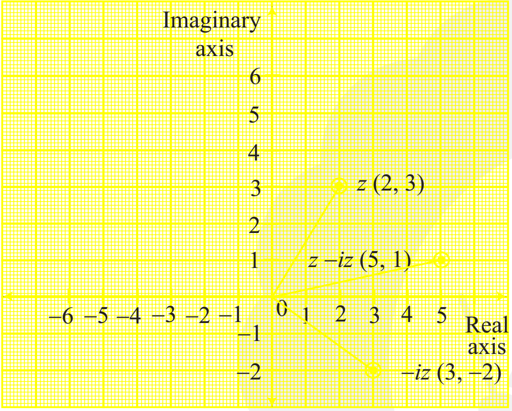
$iz = i(2 + 3i) = 2i + 3i^2 = 2i - 3 = -3 + 2i$ can be represented as (–3, 2)

$z + iz = 2 + 3i - 3 + 2i = -1 + 5i$ can be represented as (–1, 5) in the argand diagram.



(ii) $z = 2 + 3i$ can be represented as (2, 3)
$-iz = -i\ (2 + 3i) = -2i - 3i^2 = -2i - 3\ (-1)$
$= 3 - 2i$

$z - iz = 2 + 3i + 3 - 2i = 5 + i$ can be represented as (5, 1) in the Argand place.



3. Find the values of the real numbers x and y, if the complex numbers $(3 - i)x - (2 - i)y + 2i + 5$ and $2x + (-1 + 2i)$ $y + 3 + 2i$ are equal **[Hy - 2019]**

Sol. Given $(3 - i)\ x - (2 - i)\ y + 2i + 5$

$= 2x + (-1 + 2i)\ y + 3 + 2i$

$\Rightarrow 3x - ix - 2y + iy + 2i + 5 = 2x - y + 2iy + 3 + 2i$

choosing the real and imaginary parts

$(3x - 2y + 5) + i\ (-x + y + 2) = 2x - y + 3 + i\ (2y + 2)$

Equating the real and imaginary parts both sides, we get

$$3x - 2y + 5 = 2x - y + 3$$

$$\Rightarrow \quad 3x - 2y + 5 - 2x + y - 3 = 0$$

$$\Rightarrow \quad x - y = -2 \qquad \text{... (1)}$$

$$-x + y + 2 = 2y + 2$$

$$\Rightarrow \quad -x + y + 2 - 2y - 2 = 0$$

$$\Rightarrow \quad -x - y = 0 \Rightarrow x + y = 0 \qquad \text{... (2)}$$

(1) – (2) we get,

$$\begin{aligned} x - \cancel{y} &= -2 \\ x + \cancel{y} &= 0 \\ \hline 2x &= -2 \end{aligned}$$

$$\Rightarrow \quad x = -1$$

Substituting $x = -1$ in (2) we get,

$$-1 + y = 0 \Rightarrow y = 1$$

$\therefore x = -1$ and $y = 1$

EXERCISE 2.3

1. If $z_1 = 1 - 3i$, $z_2 = -4i$ and $z_3 = 5$, show that

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Sol. (i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Given $z_1 = 1 - 3i,\ z_2 = -4i$ and $z_3 = 5$

LHS $= (z_1 + z_2) + z_3$

$= [1 - 3i + (-4i)] + 5$

$= [1 - 7i] + 5 = 6 - 7i$

RHS $= z_1 + (z_2 + z_3)$

$= 1 - 3i + (-4i + 5) = 6 - 7i$

LHS = RHS

$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

LHS $= (z_1 z_2) z_3$

$= [(1 - 3i)(-4i)]\ 5 = [-4i + 12i^2]\ 5$

$= (-4i - 12)\ 5 = -20i - 60$

RHS $= z_1 (z_2 z_3)$

$= (1 - 3i)\ [(-4i)\ 5] = (1 - 3i)(-20i)$

$= -20i + 60i^2 = -20i - 60$

LHS = RHS

$\therefore (z_1 z_2) z_3 = z_1 (z_2 z_3)$

2. If $z_1 = 3$, $z_2 = 7i$, and $z_3 = 5 + 4i$, show that

(i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

(ii) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

Sol. (i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Given $z_1 = 3,\ z_2 = -7i,\ z_3 = 5 + 4i$

LHS $= z_1 (z_2 + z_3)$

$= 3\ [-7i + 5 + 4i]$

$= 3\ [5 - 3i] = 15 - 9i$

RHS $= z_1 z_2 + z_1 z_3$

$= 3\ (-7i) + 3\ (5 + 4i)$

$= -21i + 15 + 12i$

$= -9i + 15 = 15 - 9i$

LHS = RHS

$\therefore z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(ii) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

LHS $= (z_1 + z_2) z_3$

$= (3 - 7i)(5 + 4i)$

$= 15 + 12i - 35i - 28i^2$

$= 15 - 23i + 28 = 43 - 23i$

RHS $= z_1 z_3 + z_2 z_3$

$= 3\ (5 + 4i) + (-7i)(5 + 4i)$

CHAPTER
3

THEORY OF EQUATIONS

MUST KNOW DEFINITIONS

- For the quadratic equation $ax^2 + bx + c = 0$,
  (i) $\Delta = b^2 - 4ac > 0$ iff the roots are real and distinct
  (ii) $\Delta = b^2 - 4ac < 0$ iff the equation has no real roots

Fundamental theorem of algebra :

- Every polynomial equation of degree n has at least one root in C.

Complex conjugate root theorem :

- If a complex number z_0 is a root of a polynomial equation with real co-efficients, then complex conjugate $\overline{z_0}$ is also a root.
- If $p + \sqrt{q}$ is a root of a quadratic equation then $p - \sqrt{q}$ is also a root of the same equation where p. q are rational and $\sqrt{q}$ is irrational.
- If $\sqrt{p} + \sqrt{q}$ is a root of a polynomial equation then $\sqrt{p} - \sqrt{q}$, $-\sqrt{p} + \sqrt{q}$, and $-\sqrt{p} - \sqrt{q}$ are also roots of the same equation.
- If the sum of the co-efficients in $p(x) = 0$. Then 1 is a root of $p(x)$.
- If the sum of the co-efficients of odd powers = sum of the co-efficients of even powers, then –1 is a root of $p(x)$.

Rational root theorem :

- Let $a_n x^n + ...+ a_1x + a_0$ with $a_n \neq 0$, $a_0 \neq 0$ be a polynomial with integer co-efficients. If $\frac{p}{q}$ with $(p, q) = 1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n

Reciprocal polynomial :

- A polynomial $p(x)$ of degree n is said to be a reciprocal polynomial if one of the conditions is true
  (i) $p(x) = x^n \; p\left(\frac{1}{x}\right)$ (ii) $p(x) = -x^n. \; p\left(\frac{1}{x}\right)$
- A change of sign in the co-efficients is said to occur at the j^{th} power of x in $p(x)$ if the co-efficient of x^{j+1} and the co-efficient of x^j (or) co-efficient. of x^{j-1}, the co-efficient of x^j are of different signs.

IMPORTANT FORMULAE TO REMEMBER

- Vieta's formula for quadratic equation

  If α, β are the roots of $ax^2 + bx + c = 0$ then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

  Also, $x^2 - x$ (sum of the roots) + product of the roots = 0
- Vieta's formula for polynomial of degree 3.

  Co-efficient. of $x^2 = -(\alpha + \beta + \gamma)$ where α , β, γ are its roots

  Co-efficient of $x = \alpha\beta + \beta\gamma + \gamma\alpha$ and constant term $= -\alpha\beta\gamma$
- Vieta's formula for polynomial equation of degree $n > 3$

  Co-efficient of $x^{n-1} = \Sigma_1 = -\Sigma\alpha_1$

  Co-efficient of $x^{n-2} = \Sigma_2 = -\Sigma\ \alpha_1\alpha_2$

  Co-efficient of $x^{n-3} = \Sigma_3 = -\Sigma\ \alpha_1\alpha_2\alpha_3$

  Co-efficient of $x = \Sigma_{n-1} = (-1)^{n-1}\ \Sigma\ \alpha_1\alpha_2 \ldots\ldots \alpha_{n-1}$

  Co-efficient of x^0 = constant term $= \Sigma_n = (-1)^n\ \alpha_1\alpha_2\ldots\ldots\alpha_{n.}$

  A polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0 = 0$ $(a_n \neq 0)$ is a reciprocal equation iff one of the following statements is true.

  (i) $a_n = a_0,\ a_{n-1} = a_1,\ a_{n-2} = a_2, \ldots\ldots$

  (ii) $a_n = -a_0,\ a_{n-1} = -a_1,\ a_{n-2} = -a_2, \ldots\ldots.$

Descartes rule:

- If p is the number of positive zeros of a polynomial $p(x)$ with real co-efficients and s is the number of sign changes in co-efficient of $p(x)$, then $s - p$ is a non negative even integer

EXERCISE 3.1

1. **If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.** **[Aug. - 2021]**

Sol. Let x be the side and V be the volume of the cube, then volume of the cube

$$V = x \times x \times x = x^3$$

If the sides are increased by 1, 2, 3 units, Volume of the cuboid is equal to volume of the cube increased by 52

$$V + 52 = (x + 1) \times (x + 2) \times (x + 3)$$

If α, β, and γ are the roots, then If $\alpha = -1, \beta = -2$ and $\gamma = -3$.

$$\begin{aligned} \Sigma_1 &= (\alpha + \beta + \gamma) \\ &= -1 - 2 - 3 = -6 \\ \Sigma_2 &= (\alpha\beta + \alpha\gamma + \beta\gamma) \\ &= (-1)(-2) + (-1)(-3) + (-2)(-3) \\ &= 2 + 3 + 6 = 11 \\ \Sigma_3 &= (\alpha\beta\gamma) = (-1)(-2)(-3) = -6 \end{aligned}$$

Hence, the volume of the cuboid equation is

$$\begin{aligned} x^3 - \Sigma_1x^2 + \Sigma_2x - \Sigma^3 &= x^3 - (-6)x^2 + (11)x - (-6) \\ &= x^3 + 6x^2 + 11x + 6 \\ \therefore V + 52 &= x^3 + 6x^2 + 11x + 6 \\ x^3 + 52 &= x^3 + 6x^2 + 11x + 6 \end{aligned}$$

$$x^3 + 6x^2 + 11x + 6 - x^3 - 52 = 0$$

$$\begin{aligned} 6x^2 + 11x - 46 &= 0 \\ (6x + 23)(x - 2) &= 0 \\ 6x + 23 = 0 \text{ or } x - 2 &= 0 \\ 6x + 23 &= 0 \text{ gives,} \\ 6x &= -23 \\ x &= -\frac{23}{6}, \text{ is impossible} \\ \text{So } x - 2 &= 0 \text{ gives,} \\ x &= 2 \end{aligned}$$

Substituting $x = 2$, volume of the cuboid

$$\begin{aligned} V &= (2 + 1) \times (2 + 2) \times (2 + 3) \\ &= (3) \times (4) \times (5) \\ &= 60 \text{ cubic units.} \end{aligned}$$

2. Construct a cubic equation with roots

(i) 1, 2, and 3 (ii) 1, 1, and −2

(iii) 2, $\frac{1}{2}$ and 1

Sol. (i) 1,2, and 3

Given roots are 1,2 and 3

Here α = 1, β = 2 and γ = 3

A cubic polynomial equation whose roots are α, β, γ is

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\,x - \alpha\,\beta\,\gamma = 0$

$\Rightarrow\ x^3 - (1 + 2 + 3)\,x^2 + (2 + 6 + 3)\,x - 6 = 0$

$\Rightarrow\ x^3 - 6x^2 + 11x - 6 = 0.$

(ii) Here α = 1, β = 1 and γ = –2

∴ The required cubic equation is

$x^3 - (1 + 1 - 2)\,x^2 + (1 - 2 - 2)\,x - (1)(1)(-2) = 0$

$x^3 - 0x^2 - 3x + 2 = 0$

$x^3 - 3x + 2 = 0.$

(iii) Here α = 2, β = $\frac{1}{2}$ and γ = 1

∴ The cubic equation is

$$x^3 - \left(2 + \frac{1}{2} + 1\right)x^2 + \left(1 + \frac{1}{2} + 2\right)x - 1 = 0$$

$\Rightarrow\ 2x^3 - 7x^2 + 7x - 2 = 0$

3. If α, β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are

(i) 2α, 2β, 2γ, [Hy - 2019] (ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$,

(iii) –α, – β, –γ

Sol. The roots of $x^3 + 2x^2 + 3x + 4 = 0$ are α, β, γ

$\therefore\ \alpha + \beta + \gamma = -$ co-efficient of $x^2 = -2$ ...(1)

$\alpha\beta + \beta\gamma + \gamma\alpha =$ co-efficient of $x = 3$ ...(2)

$-\alpha\,\beta\,\gamma = +4 \Rightarrow \alpha\,\beta\,\gamma = -4$ ...(3)

(i) Form a cubic equation whose roots are 2α, 2β, 2γ

$2\alpha + 2\beta + 2\gamma = 2\,(\alpha + \beta + \gamma) = 2\,(-2) = -4$ [from (1)]

$4\alpha\beta + 4\beta\gamma + 4\gamma\alpha = 4\,(\alpha\beta + \beta\gamma + \gamma\alpha) = 4(3) = 12$ [from (2)]

$(2\alpha)\ (2\beta)\ (2\gamma) = 8\,(\alpha\,\beta\,\gamma) = 8(-4) = -32$ [from (3)]

∴ The required cubic equation is

$x^3 - (2\alpha + 2\beta + 2\gamma)\,x^2 + (4\alpha\beta + 4\beta\gamma + 4\gamma\alpha)\,x - (2\alpha)\ (2\beta)\ (2\gamma) = 0$

$\Rightarrow\ x^3 - (-4)x^2 + 12x + 32 = 0$

$\Rightarrow\ x^3 + 4x^2 + 12x + 32 = 0$

(ii) Form the cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{-4} = \frac{-3}{4}$$

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \frac{-2}{-4} = \frac{1}{2}$$

$$\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma} = \frac{1}{-4} = -\frac{1}{4}$$

∴ The required cubic equation is

$$x^3 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)x^2 + \left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)x - \left(\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}\right) = 0$$

$$\Rightarrow x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = 0$$

Multiplying by 4 we get,

$4x^3 + 3x^2 + 2x + 1 = 0$

(iii) Form the equation whose roots are α – β – γ

$$\therefore -\alpha - \beta - \gamma = -\left(\alpha + \beta + \gamma\right) = -(-2) = 2$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3$$

$$(-\alpha)(-\beta)(-\gamma) = -\left(\alpha\beta\gamma\right) = -(-4) = 4$$

∴ The required cubic equation is

$$x^3 - \left(-\alpha - \beta - \gamma\right)x^2 + \left(\alpha\beta + \beta\gamma + \gamma\alpha\right)x - \left[(-\alpha)(-\beta)(-\gamma)\right] = 0$$

$\Rightarrow\ x^3 - (2)x^2 + 3x - 4 = 0$

$\Rightarrow\ x^3 - 2x^2 + 3x - 4 = 0$

4. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.

Sol. Given cubic equation is $3x^3 - 16x^2 + 23x - 6 = 0$

Let $\alpha, \frac{1}{\alpha}$ and γ be the roots of the equation

[∵ product of two roots is 1]

$$x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - 2 = 0 \qquad ...(1)$$

Comparing (1) with

$$x^3 - \left(\frac{\alpha + 1 + \gamma}{\alpha}\right)x^2 + \left(\alpha\frac{1}{\alpha} + \frac{1}{\alpha}\cdot\gamma + \gamma\alpha\right)^{x} - \alpha\frac{1}{\alpha}\cdot\gamma = 0 \qquad ...(2)$$

we get,

$$\alpha + \frac{1}{\alpha} + \gamma = \frac{16}{3} \qquad ...(3)$$

$$1 + \frac{\gamma}{\alpha} + \gamma\alpha = \frac{23}{3}$$

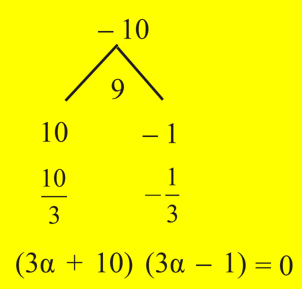
$$\alpha\cdot\frac{1}{\alpha}\cdot\gamma = 2 \Rightarrow \gamma = 2 \qquad ...(4)$$

Substituting γ = 2 in (3)

$$\alpha+\frac{1}{\alpha}+2 = \frac{16}{3}$$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha} = \frac{16}{3}-2 = \frac{16-6}{3} = \frac{10}{3}$$

$$\Rightarrow \quad \frac{\alpha^2+1}{\alpha} = \frac{10}{3}$$



(3α + 10) (3α – 1) = 0

$$3\alpha^2+3 = 10\alpha$$

$$3\alpha^2-10\alpha+3 = 0$$

$$(3\alpha+10)(3\alpha-1) = 0$$

$$\alpha = \frac{-10}{3} \text{ (or)}$$

$$\alpha = \frac{1}{3}$$

$$\alpha = \frac{-10}{3} \text{ is not possible } \Rightarrow \alpha = \frac{1}{3}$$

$$\left[\because \alpha = \frac{-10}{3} \text{ will not satisfy (5)}\right]$$

∴ The roots are 3, $\frac{1}{3}$, 2.

5. **Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$.** **[PTA - 2]**

Sol. Given equation is $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Here $a = 2$, $b = -8$, $c = 6$, $d = 0$, $e = -3$

Let α, β, γ and δ be the roots of equation (1)

Then by Vieta's formula,

$$\Sigma_1 = \alpha+\beta+\gamma+\delta = \frac{-b}{a} = \frac{-(-8)}{2} = 4$$

$$\Sigma_2 = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a} = \frac{6}{2} = 3$$

$$\Sigma_3 = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = \frac{-d}{a} = \frac{0}{2} = 0$$

$$\Sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a} = \frac{-3}{2}$$

Now, $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$

$$\Rightarrow \quad \alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = 4^2 - 2(3) = 16 - 6 = 10$$

6. **Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the ratio 3:2.**

Sol. Given $x^3 - 9x^2 + 14x + 24 = 0$

Two of its roots are in the ratio 3 : 2

$$\text{Let } \alpha : \beta = 3 : 2$$

$$\frac{\alpha}{\beta} = \frac{3}{2}$$

$$\therefore 2\alpha = 3\beta$$

Comparing with $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$$\Sigma_1 = (\alpha+\beta+\gamma) = 9$$

$$\Sigma_2 = (\alpha\beta+\alpha\gamma+\beta\gamma) = 14$$

$$\Sigma_3 = (\alpha\beta\gamma) = -24$$

$$\text{From } \alpha+\beta+\gamma = 9$$

$$2\alpha+2\beta+2\gamma = 18$$

Substituting $2\alpha = 3\beta$

$$3\beta+2\beta+2\gamma = 18$$

$$5\beta+2\gamma = 18$$

$$2\gamma = 18-5\beta \qquad \text{... (1)}$$

$$\text{From } \alpha\beta\gamma = -24$$

$$2\alpha\beta\gamma = -48$$

Substituting $2\alpha = 3\beta$

$$(3\beta)\beta\gamma = -48$$

$$3\beta^2\gamma = -48$$

$$\beta^2\gamma = -16$$

Multiplying by 2, $\beta^2 2\gamma = -32$

Substituting $2\gamma = 18 - 5\beta$

$$\beta^2(18-5\beta) = -32$$

$$18\beta^2-5\beta^3 = -32$$

$$5\beta^3-18\beta^2-32 = 0$$

$$(\beta-4)(5\beta^2+2\beta+8) = 0$$

$\beta - 4 = 0$, gives $\beta = 4$

Substituting $\beta = 4$ in $2\alpha = 3\beta$

$$2\alpha = 3(4)$$

$$2\alpha = 12$$

$$\therefore \alpha = 6$$

Substituting $\beta = 4$ in $2\gamma = 18 - 5\beta$

$$2\gamma = 18-5(4)$$

$$2\gamma = 18-20$$

$$2\gamma = -2$$

$$\therefore \gamma = -1$$

Hence the roots are 6, 4 and −1

7. **If α, β and γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.**

Sol. Given α, β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$

$$\therefore \alpha+\beta+\gamma = \frac{-b}{a}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\text{Now, } \sum\frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta}$$

$$= \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}$$

$$= \frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{-\frac{d}{a}}$$

$$= \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{-\frac{d}{a}} \Rightarrow \frac{b^2-2ac}{a^2}\times\frac{-a}{d}$$

$$\therefore \sum\frac{\alpha}{\beta\gamma} = -\frac{\left(b^2-2ac\right)}{ad} = \frac{2ac-b^2}{ad}$$

8. If α, β, γ and δ are the roots of the polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a quadratic equation with integer coefficients whose roots are α + β + γ + δ and α β γ δ

Sol. Given polynomial equation is **[Sep - 2020]**

$$2x^4+5x^3-7x^2+8 = 0$$

Here $a = 2$, $b = 5$, $c = -7$, $d = 0$, $e = 8$

By Vieta's formula,

$$\alpha+\beta+\gamma+\delta = \frac{-b}{a} = \frac{-5}{2}$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}=\frac{-7}{2}$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = \frac{-d}{a}=0$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{8}{2} = 4$$

Given roots of the quadratic equation are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$

$$\therefore \text{sum of the roots} = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$$

$$= \left(\frac{-5}{2}+4\right) = \frac{-5+8}{2} = \frac{3}{2}$$

$$\text{Product of the roots} = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$$

$$= \left(\frac{-5}{2}\right)(4) = \frac{-20}{2} = -10$$

∴ The required quadratic equation is $x^2 - x$ (sum of the roots) + product of the roots = 0

$$\Rightarrow x^2 - x\left(\frac{3}{2}\right) - 10 = 0$$

$$\Rightarrow 2x^2-3x-20 = 0$$

9. If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}} = 0$.

Sol. Given p, q are the roots of $lx^2 + nx + n = 0$

$$p+q = -\frac{n}{l} \text{ and } pq = \frac{n}{l}$$

$$\text{Consider } \frac{(p+q)^2}{pq} = \frac{\left(\frac{-n}{l}\right)^2}{\left(\frac{n}{l}\right)} = \frac{n^2}{l^2}\times\frac{l}{n} = \frac{n}{l}$$

Taking square root on both sider

$$\sqrt{\frac{(p+q)^2}{pq}} = \sqrt{\frac{n}{l}}$$

$$\Rightarrow \pm\frac{(p+q)}{\sqrt{pq}} = \sqrt{\frac{n}{l}}$$

$$\text{Consider } -\frac{(p+q)}{\sqrt{pq}} = \sqrt{\frac{n}{l}}$$

$$\Rightarrow \frac{(p+q)}{\sqrt{pq}} = -\sqrt{\frac{n}{l}}$$

$$\frac{p}{\sqrt{pq}}+\frac{q}{\sqrt{pq}} = -\sqrt{\frac{n}{l}}$$

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}} = 0$$

Hence proved.

10. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

Sol. Given equation are $x^2+px+q = 0$ ...(1)

and $x^2+p'x+q' = 0$ ...(2)

Let α be the common root for (1) and (2)

$\therefore \alpha^2+p\alpha+q = 0$ ...(3)

and $\alpha^2+p'\alpha+q' = 0$ ...(4)

Solving (3) and (4) by cross multiplication method we get

$$\begin{array}{cccccccc} p & q & 1 & p & & & \\ p' & q' & 1 & p' & \alpha^2 & \alpha & 1 \end{array}$$

$$\Rightarrow \quad \frac{\alpha^2}{pq'-p'q} = \frac{\alpha}{q-q'} = \frac{1}{p'-p}$$

Consider $\dfrac{\alpha^2}{pq'-p'q} = \dfrac{\alpha}{q-q'} \Rightarrow \dfrac{\alpha^2}{\alpha} = \dfrac{pq'-p'q}{q-q'}$

$$\alpha = \frac{pq'-p'q}{q-q'}$$

Consider $\dfrac{\alpha}{q-q'} = \dfrac{1}{p'-p} \Rightarrow \alpha = \dfrac{q-q'}{p'-p}$

Hence its roots are $\dfrac{pq'-p'q}{q-q'}$ or $\dfrac{q-q'}{p'-p}$

11. **A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was left standing.**

Sol. Given that the height of the tree is 12m.

Let x m be the standing part and $(12 - x)$m be the broken part.

Given $\quad x = \sqrt[3]{12-x} \Rightarrow x = (12-x)^{\frac{1}{3}}$

Taking power 3 both sides, we get

$\Rightarrow \quad x^3 = 12 - x$

$\Rightarrow x^3 + x - 12 = 0$ which is the required mathematical problem.

EXERCISE 3.2

1. **If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, in terms of k.**

Sol. Given equation is $2x^2 + kx + k = 0$

Here $a = 2, b = k, c = k$

$\therefore$ Discriminant

$$\Delta = b^2 - 4ac = k^2 - 4\,(2)(k)$$
$$= k^2 - 8k = k\,(k-8)$$

Since k is real, the possible values of k are

$k < 0, k = 0$ or $8, 0 < k < 8$ or $k > 8$

Case (i) When $k < 0, \Delta = k\,(k - 8) > 0$

$\Rightarrow$ it has real roots ($\because k$ is real)

Case (ii) When $k = 0$ or $8, \Delta = 0$,

$\therefore$ The roots are real and equal

Case (iii) When $0 < k < 8, \Delta = k\,(k - 8) < 0$

It has imaginary roots

Case (iv) When $k > 8, \Delta = k\,(k - 8) > 0$

$\Rightarrow$ The roots are real and distinct

2. **Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}i$ as a root.**

Sol. Given $2 + \sqrt{3}\,i$ is a root

$\therefore 2 - \sqrt{3}\,i$ is another root

Hence sum of the roots $= 2 + \sqrt{3}i + 2 - \sqrt{3}i = 4$

Product of the roots $= (2 + \sqrt{3}\,i)(2 - \sqrt{3}\,i)$

$= 4 + 3 = 7$

Hence the required equatoin is

x^2 – (sum of the roots) x + product of the roots = 0

$$x^2 - 4x + 7 = 0$$

3. **Find a polynomial equation of minimum degree with rational coefficients, having $2i + 3$ as a root.** **[Hy - 2019]**

Sol. Given $2i + 3$ is a root

$\therefore$ Its conjugate $3 - 2i$ is also a root of the polynomial equation.

$\therefore$ Sum of the roots $= \cancel{2i} + 3 + 3 - \cancel{2i} = 6$

Product of the roots $= (3 + 2i)\,(3 - 2i)$

$= 3^2 + 2^2 = 9 + 4 = 13$

$\therefore$ The polynomial equation of minimum degree with rational co-efficients is

$x^2 - x$ (sum of the roots) + product of the roots = 0

$\Rightarrow \quad x^2 - x\,(6) + 13 = 0$

$\Rightarrow \quad x^2 - 6x + 13 = 0$

4. **Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root.** **[PTA - 5]**

Sol. Given $(\sqrt{5} - \sqrt{3})$ is a root

$\Rightarrow \sqrt{5} + \sqrt{3}$ is also a root.

$\therefore$ Sum of the roots $= \sqrt{5} - \sqrt{3} + \sqrt{5} + \sqrt{3} = 2\sqrt{5}$

Product of the roots

$= (\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

$= (\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3 = 2$

$\therefore$One of the factor is $x^2 - x$ (sum of the roots) + product of the roots

$\Rightarrow x^2 - 2x\sqrt{5} + 2$

The other factor also will be $x^2 + 2x\sqrt{5} + 2$.

Hence, the polynomial equation of minimum degree with rational co-efficients is

$$(x^2 - 2x\sqrt{5} + 2)\,(x^2 + 2x\sqrt{5} + 2) = 0$$

$$\Rightarrow \quad (x^2 + 2 - 2\sqrt{5}\,x)(x^2 + 2 + 2\sqrt{5}\,x) = 0$$

$$\Rightarrow \quad (x^2 + 2)^2 - (2\sqrt{5}\,x)^2 = 0$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \quad x^4 + 4x^2 + 4 - 4(5)x^2 = 0$$

$$\Rightarrow \quad x^4 + 4x^2 + 4 - 20x^2 = 0$$

$$\Rightarrow \quad x^4 - 16x^2 + 4 = 0$$

5. Prove that a straight line and parabola cannot intersect at more than two points.

Sol. By choosing the co-ordinate axes suitably, we take the equation of the straight line as

$$y = mx + c \qquad ...(1)$$

and equation of parabola as $y^2 = 4ax$ ...(2)

Substituting (1) in (2), we get

$$(mx + c)^2 = 4ax$$

$$\Rightarrow \quad m^2x^2 + c^2 + 2mcx = 4ax$$

$$\Rightarrow \quad m^2x^2 + x(2mc - 4a) + c^2 = 0$$

Which is a quadratic equation in x.

This equation cannot have more than two solution. Hence, a straight line and a parabola cannot intersect at more than two points.

EXERCISE 3.3

1. Solve the cubic equation : $2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.

Sol. Since sum of two of its roots vanishes, let the roots be α, – α and β

$$\alpha - \alpha + \beta = \frac{-b}{a} = \frac{1}{2} \Rightarrow \beta = \frac{1}{2}$$

$$\text{Also, } \alpha\beta\gamma = \frac{-d}{a} \Rightarrow \alpha(-\alpha)\left(\frac{1}{2}\right) = \frac{-9}{2}$$

$$\Rightarrow \quad \frac{\alpha^2}{2} = \frac{9}{2} \Rightarrow \alpha^2 = 9 \Rightarrow \alpha = \pm 3$$

$$\Rightarrow \quad \alpha = 3$$

∴ The roots are 3, –3 and $\frac{1}{2}$.

2. Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progression.

Sol. Here, $a = 9$, $b = -36$, $c = 44$, $d = -16$

Since the roots form an arithmetic progression,

Let the roots be $a - d$, a and $a + d$

$$\text{Sum of the roots} = \frac{-b}{a}$$

$$\Rightarrow (a - d) + (a) + (a + d) = \frac{-(-36)}{9} = 4$$

$$\Rightarrow \quad 3a = 4 \Rightarrow a = \frac{4}{3}$$

$$\text{and product of the roots} = \frac{-d}{a}$$

$$= \frac{-(-16)}{9} = \frac{16}{9}$$

$$\Rightarrow \quad (a - d)(a)(a + d) = \frac{16}{9}$$

$$(a^2 - d^2)(a) = \frac{16}{9}$$

$$\left(\frac{16}{9} - d^2\right)\left(\frac{4}{3}\right) = \frac{16}{9} \qquad \left[\because a = \frac{4}{3}\right]$$

$$\frac{16}{9} - d^2 = \frac{16}{9} \times \frac{3}{4} = \frac{4}{3}$$

$$\frac{16}{9} - \frac{4}{3} = d^2$$

$$\Rightarrow \quad d^2 = \frac{16 - 12}{9} = \frac{4}{9}$$

$$\Rightarrow \quad d = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$$

∴The roots are $a - d$, a, $a + d$

(i) When $a = \frac{4}{3}$, and $d = \frac{2}{3}$.

The roots $a - d$, a, $a + d$ becomes,

$$\frac{4}{3} - \frac{2}{3}, \frac{4}{3}, \frac{4}{3} + \frac{2}{3} \Rightarrow \frac{2}{3}, \frac{4}{3}, 2.$$

(ii) When $a = \frac{4}{3}$, and $d = -\frac{2}{3}$.

The roots $a - d$, a, $a + d$ becomes,

$$\frac{4}{3} + \frac{2}{3}, \frac{4}{3}, \frac{4}{3} - \frac{2}{3} \Rightarrow 2, \frac{4}{3}, \frac{2}{3}.$$

3. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if its roots form a geometric progression.

Sol. Given cubic equation is $3x^3 - 26x^2 + 52x - 24 = 0$

Here, $a = 3$, $b = -26$, $c = 52$, $d = -24$.

Since the roots form a geometric progression,

The roots are $\frac{a}{r}$, a, ar. sum of the roots $= \frac{-b}{a}$

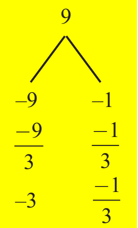
$$\Rightarrow \quad \frac{a}{r} + a + ar = \frac{26}{3} \qquad ...(1)$$

and product of the roots $= \frac{-d}{a}$

$$\Rightarrow \quad \frac{a}{\cancel{r}} \times a \times a\cancel{r} = \frac{24}{3} = 8$$

$$a^3 = 8 = 2^3$$

$$a = 2$$



∴ (1) becomes $\frac{2}{r}+2+2r=\frac{26}{3}$

$$\Rightarrow \frac{2+2r+2r^2}{r} = \frac{26}{3} \Rightarrow \frac{1+r+r^2}{r}=\frac{13}{3}$$

$$\Rightarrow \quad 3+3r+3r^2 = 13r \Rightarrow 3r^2-10r+3 = 0$$

$$\Rightarrow \quad (r-3)(3r-1) = 0 \Rightarrow r = 3$$

∴The roots are $\frac{a}{r}$, a, ar

$$\Rightarrow \frac{2}{3}, 2, 2(3) = \frac{2}{3}, 2, 6$$

4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.

Sol. Given cubic equation is $2x^3 - 6x^2 + 3x + k = 0$

Here, $a = 2$, $b = -6$, $c = 3$, $d = k$

Let α, β, γ be the roots

$$\text{Given } \alpha = 2(\beta+\gamma) \Rightarrow \frac{\alpha}{2} = \beta+\gamma \quad \text{...(1)}$$

$$\text{Now, } \alpha+\beta+\gamma = \frac{-b}{a} = -\frac{(-6)}{2} = 3$$

$$\frac{\alpha}{2}+\alpha = 3 \Rightarrow \frac{\alpha+2\alpha}{2} = 3 \Rightarrow \frac{3\alpha}{2} = 3$$

$$\Rightarrow \qquad \alpha = 2$$

$$\alpha\beta\gamma = \frac{-d}{a} = \frac{-k}{2} \Rightarrow 2.\beta\gamma = \frac{-k}{2}$$

$$\beta\gamma = \frac{-k}{4} \qquad \text{...(2)}$$

$$\text{Also, } \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$$

$$2\beta+\beta\gamma+2\gamma = \frac{3}{2}$$

$$2(\beta+\gamma)+\beta\gamma = \frac{3}{2}$$

$$\alpha-\frac{k}{4} = \frac{3}{2} \qquad \text{[From (1) \& (2)]}$$

$$2-\frac{k}{4} = \frac{3}{2} \qquad [\because \alpha = 2]$$

$$2-\frac{3}{2} = \frac{k}{4} \Rightarrow \frac{1}{2} = \frac{k}{4}$$

$$k = \frac{4}{2} \Rightarrow k = 2$$

$$\text{From (2), } \beta\gamma = \frac{-k}{4} = \frac{-2}{4} = \frac{-1}{2} \Rightarrow \gamma = \frac{-1}{2\beta}$$

$$\text{From (1), } \beta+\gamma = \frac{\alpha}{2} = \frac{2}{2} = 1$$

Substituting $\gamma = \frac{-1}{2\beta}$ we get

$$\beta-\frac{1}{2\beta} = 1 \Rightarrow 2\beta^2-1 = 2\beta \Rightarrow 2\beta^2-2\beta-1 = 0$$

$$\beta = \frac{2\pm\sqrt{4-4(2)(-1)}}{4} = \frac{2\pm\sqrt{4+8}}{4}$$

$$= \frac{2\pm\sqrt{12}}{4} = \frac{2\pm2\sqrt{3}}{4}$$

$$\beta = \frac{1\pm\sqrt{3}}{2}$$

$k = 2$ and the roots are 2, $\frac{(1\pm\sqrt{3})}{2}$

5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

Sol. Let $f(x) = x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$

Given $(1 + 2i)$ is a root ⇒ $(1 - 2i)$ is also a root.

Also $\sqrt{3}$ is a root ⇒ $-\sqrt{3}$ is also a root.

Hence, the factors of $f(x)$ are $[x - (1 + 2i)]$

$$[x-(1-2i)]\,[x-\sqrt{3}\,][x+\sqrt{3}\,]$$

$$[(x-1)-2i]\,[(x-1)+2i][x-\sqrt{3}\,][x+\sqrt{3}\,]$$

$$\left((x-1)^2+2^2\right)\left(x^2-3\right)$$

$$= \left(x^2-2x+1+4\right)\left(x^2-3\right)$$

⇒ factor of $f(x)$ is $(x^2 - 2x + 5)(x^2 - 3)$

$$\Rightarrow x^4-3x^2-2x^3+6x+5x^2-15$$

⇒$(x^4 - 2x^3 + 2x^2 + 6x - 15)$ is a factor of $f(x)$

To find the other factor, let us divide $f(x)$ by $x^4 - 2x^3 + 2x^2 + 6x - 15$

$$\begin{array}{r|l}
 & x^2 - x \;- 9 \\ \hline
x^4-2x^3+2x^2+6x-15 & \cancel{x^6}-3x^5-5x^4+22x^3-39x^2-39x+135 \\
 & (-)\;(+)\;(-)\;(-)\;(+) \\
 & \cancel{x^6}-2x^5+2x^4+6x^3-15x^2 \\ \hline
 & -\cancel{x^5}-7x^4+16x^3-24x^2-39x \\
 & (+)\;(-)\;(+)\;(+)\;(-) \\
 & -x^5+2x^4-2x^3-6x^2+15x \\ \hline
 & -9\cancel{x^4}+18\cancel{x^3}-18\cancel{x^2}-54\cancel{x}+13\cancel{5} \\
 & (+)\;(-)\;(+)\;(+)\;(-) \\
 & -\cancel{9}x^4+\cancel{1}8x^3-\cancel{1}8x^2-\cancel{5}4x+\cancel{1}35 \\ \hline
 & \quad 0
\end{array}$$

The other factor is $x^2 - x - 9$

$$\Rightarrow x = \frac{1\pm\sqrt{(-1)^2-4(1)(-9)}}{2}$$

$$\left[\because x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{37}}{2}$$

Hence the roots are

$1 + 2i,\ 1-2i,\ \sqrt{3}, -\sqrt{3}, \dfrac{1+\sqrt{37}}{2}, \dfrac{1-\sqrt{37}}{2}$

6. Solve the cubic equations :

(i) $2x^3 - 9x^2 + 10x = 3$,

(ii) $8x^3 - 2x^2 - 7x + 3 = 0$.

Sol. (i) Since the sum of the co-efficients is

$2 - 9 + 10 - 3 = 12 - 12 = 0$

$\Rightarrow \quad x = 1$ is a root of $f(x)$

$\therefore (x - 1)$ is a factor of $f(x)$

To find the other factor, let us divide $f(x)$ by $x - 1$

$$\begin{array}{r|rrrr}
1 & 2 & -9 & 10 & -3 \\
 & \downarrow & 2 & -7 & 3 \\
\hline
3 & 2 & -7 & 3 & \boxed{0} \\
 & \downarrow & 6 & -3 & \\
\hline
 & 2 & -1 & \boxed{0} &
\end{array}$$

[Using synthetic division]

$f(x) = (x-1)(x-3)(2x-1) = 0$

$\Rightarrow \quad x - 1 = 0,\ x - 3 = 0$ or $2x - 1 = 0$

$\Rightarrow \quad x = 1,\ x = 3,\ x = \dfrac{1}{2}$

Hence the roots are $1, 3, \dfrac{1}{2}$.

(ii) Let $f(x) = 8x^3 - 2x^2 - 7x + 3 = 0$

Here sum of the co-efficients of odd terms

$= 8 - 7 = 1$

and sum of the co-efficients of even terms

$= -2 + 3 = 1$

Hence, $x = -1$ is a root of $f(x)$

Let us divide $f(x)$ by $(x + 1)$

$$\begin{array}{r|rrrr}
-1 & 8 & -2 & -7 & 3 \\
 & \downarrow & -8 & 10 & -3 \\
\hline
 & 8 & -10 & 3 & \boxed{0}
\end{array}$$

$\therefore$ The other factor is $8x^2 - 10x + 3$

$$\Rightarrow \quad x = \frac{10 \pm \sqrt{100 - 4(8)(3)}}{2 \times 8}$$

$$\Rightarrow \quad x = \frac{10 \pm \sqrt{100 - 96}}{16} \Rightarrow x = \frac{10 \pm 2}{16}$$

$$\Rightarrow \quad x = \frac{12}{16} \text{ or } x = \frac{8}{16} \Rightarrow x = \frac{3}{4}, \frac{1}{2}.$$

$\therefore$ The roots are $-1, \dfrac{1}{2}, \dfrac{3}{4}$

7. Solve the equation : $x^4 - 14x^2 + 45 = 0$.

$$\begin{array}{c} 45 \\ -14 \\ -9 \quad\quad -5 \end{array}$$

Sol. Put $x^2 = y$

$\Rightarrow \quad y^2 - 14y + 45 = 0$

$\Rightarrow \quad (y - 9)(y - 5) = 0$

$\Rightarrow \quad y = 9$ or $y = 5$

$\Rightarrow \quad x^2 = 9$ or $x^2 = 5$

$\Rightarrow \quad x = \pm 3$ or $x = \pm\sqrt{5}$

Hence the roots are $\pm 3, \pm\sqrt{5}$.

EXERCISE 3.4

1. Solve : (i) $(x - 5)(x - 7)(x + 6)(x + 4) = 504$

Sol. Rearrange the terms as.

$(x - 5)(x + 4)(x - 7)(x + 6) = 504$

$\Rightarrow (x^2 - x - 20)(x^2 - x - 42) = 504$

$$\begin{array}{c} 336 \\ -62 \\ -56 \quad\quad -6 \end{array}$$

put $x^2 - x = y$

$\Rightarrow \quad (y - 20)(y - 42) = 504$

$\Rightarrow y^2 - 62y + 840 - 504 = 0$

$\Rightarrow \quad y^2 - 62y + 336 = 0$

$\Rightarrow \quad (y - 56)(y - 6) = 0$

$\Rightarrow \quad y = 56, 6$

Case (i) When $y = 56$, $x^2 - x = 56$

$$\begin{array}{c} -56 \\ -1 \\ -8 \quad\quad 7 \end{array}$$

$\Rightarrow \quad x^2 - x - 56 = 0$

$\Rightarrow \quad (x - 8)(x + 7) = 0$

$\Rightarrow \quad x = 8, -7$

Case (ii)

When $y = 6$,

$x^2 - x = 6$

$$\begin{array}{c} 6 \\ -1 \\ -3 \quad\quad 2 \end{array}$$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad (x - 3)(x + 2) = 0$

$\Rightarrow \quad x = 3, -2$

Hence the roots are $\{8, -7, -3, -2\}$.

(ii) $(x - 4)(x - 7)(x - 2)(x + 1) = 16$

Sol. Given $(x - 4)(x - 7)(x - 2)(x + 1) = 16$

Re arrange the terms as

$(x - 4)(x - 2)(x - 7)(x + 1) = 16$

$\Rightarrow \quad (x^2 - 6x + 8)(x^2 - 6x - 7) = 16$

put $x^2 - 6x = y$

$$\begin{array}{c} -72 \\ -1 \\ +9 \quad\quad -8 \end{array}$$

$\Rightarrow \quad (y + 8)(y - 7) = 16$

$\Rightarrow \quad y^2 + y - 56 - 16 = 0$

$\Rightarrow \quad y^2 + y - 72 = 0$

$\Rightarrow \quad (y + 9)(y - 8) = 0$

$\Rightarrow \quad y = -9, 8$

Case (i)

When $y = -9$

$x^2 - 6x = -9$

$\Rightarrow \quad x^2 - 6x + 9 = 0$

$\Rightarrow \quad (x - 3)^2 = 0$

$\Rightarrow \quad x = 3, 3$

Case (ii)

When $y = 8,$

$x^2 - 6x = 8$

$\Rightarrow \quad x^2 - 6x - 8 = 0$

$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36 - 4(1)(-8)}}{2}$$

$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36 + 32}}{2}$$

$$\Rightarrow \quad x = \frac{6 \pm \sqrt{68}}{2} \Rightarrow x = \frac{6 \pm 2\sqrt{17}}{2}$$

$$\Rightarrow \quad x = \frac{2\left(3 \pm \sqrt{17}\right)}{2}$$

$$\Rightarrow \quad x = 3 \pm \sqrt{17}$$

–72

–1

–9 –8

$\therefore$ The roots are 3, 3, 3 + $\sqrt{17}$ and 3 – $\sqrt{17}$

2. **Solve : $(2x - 1)(x + 3)(x - 2)(2x + 3) + 20 = 0$**

Sol. Rearrange the terms as

$(2x - 1)\ (2x + 3)\ (x + 3)\ (x - 2) + 20 = 0$

$\Rightarrow (4x^2 + 6x - 2x - 3)\ (x^2 - 2x + 3x - 6) + 20 = 0$

$\Rightarrow (4x^2 + 4x - 3)\ (x^2 + x - 6) + 20 = 0$

put $x^2 + x = y$

$\Rightarrow (4y - 3)\ (y - 6) + 20 = 0$

$\Rightarrow 4y^2 - 24y - 3y + 18 + 20 = 0$

$\Rightarrow 4y^2 - 27y + 38 = 0$

$\Rightarrow (y - 2)\ (4y - 19) = 0$

$$\Rightarrow \quad y = 2, \frac{19}{4}$$

152

–27

–19 –8

$\frac{-19}{4}$ $\frac{-8}{4}$

$\frac{-19}{4}$ –2

Case (i)

When $y = 2,$

$x^2 + x = 2 \Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x + 2)\ (x - 1) = 0$

$\Rightarrow \quad x = -2, 1$

Case (ii)

When $y = \frac{19}{4}, \quad x^2 + x = \frac{19}{4}$

$\Rightarrow \quad 4x^2 + 4x = 19$

$\Rightarrow \quad 4x^2 + 4x - 19 = 0$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16 - 4(4)(-19)}}{8}$$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16 + 304}}{8}$$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{320}}{8}$$

$$\Rightarrow \quad x = \frac{-4 \pm 8\sqrt{5}}{8}$$

$$\Rightarrow \quad x = \frac{4\left(-1 \pm 2\sqrt{5}\right)}{8}$$

$$\Rightarrow \quad x = \frac{-1 \pm 2\sqrt{5}}{2}$$

$$\therefore \text{ The roots are } \left\{1, -2, \frac{-1 + 2\sqrt{5}}{2}, \frac{-1 - 2\sqrt{5}}{2}\right\}$$

EXERCISE 3.5

1. **Solve the following equations**

(i) $\sin^2 x - 5 \sin x + 4 = 0$

(ii) $12x^3 + 8x = 29x^2 - 4$

Sol. **(i) $\sin^2 x - 5 \sin x + 4 = 0$**

Put $y = \sin x$

$\Rightarrow \quad y^2 - 5y + 4 = 0$

$\Rightarrow \quad (y - 4)(y - 1) = 0$

$\Rightarrow \quad y = 4, 1$

Case (i)

When $y = 4, \sin x = 4$

and no solution for $\sin x = 4$ since the range the sine function is $[-1, 1]$

Case (ii)

When $y = 1, \sin x = 1$

$$\Rightarrow \quad \sin x = \sin \frac{\pi}{2} \qquad \left[\because \sin \frac{\pi}{2} = 1\right]$$

$$\Rightarrow \quad x = n\pi + (-1)^n \frac{\pi}{2}, n \in \mathbb{Z}$$

$[\because \sin x = \sin \alpha \Rightarrow x = n\pi + (-1)^n\ \alpha, n \in \mathbb{Z}]$

for all $n \in \mathbb{Z}$, $x = 2n + \pi\ \frac{n}{2}$ is solution

(ii) $12x^3 + 8x = 29x^2 - 4$

This equation can be re-written as

$12x^3 - 29x^2 + 8x + 4 = 0$

$$\begin{array}{r|rrrr} +2 & 12 & -29 & 8 & 4 \\ & \downarrow & +24 & -10 & -4 \\ \hline & 12 & -5 & -2 & \boxed{0} \end{array}$$

$\therefore x = 2$ is a root and the remaining factor is

$12x^2 - 5x - 2$

$\Rightarrow \quad (3x - 2)(4x + 1) = 0$

$\Rightarrow \quad 3x - 2 = 0 \text{ or } 4x + 1 = 0$

$$\Rightarrow \quad x = \frac{2}{3}, x = \frac{-1}{4}$$

$$\therefore \quad \text{The roots are } 2, \frac{2}{3} \text{ and } -\frac{1}{4}.$$

–24

–5

–8 3

$\frac{-8}{12}$ $\frac{3}{12}$

$\frac{-2}{3}$ $\frac{1}{4}$

2. Examine for the rational roots of

(i) $2x^3 - x^2 - 1 = 0$ **(ii)** $x^8 - 3x + 1 = 0$

Sol. **(i)** $2x^3 - x^2 - 1 = 0$

Since the sum of the co-efficients = 2 – 1 – 1 = 0

$x = 1$ is a root.

$$\begin{array}{c|cccc} 1 & 2 & -1 & 0 & -1 \\ & \downarrow & 2 & 1 & 1 \\ \hline & 2 & 1 & 1 & \boxed{0} \end{array}$$

$\therefore x = 1$ is a root and the remaining factor is $2x^2 + x + 1$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1^2 - 4(2)(1)}}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-7}}{2} \text{ which is a complex root.}$$

$$\Rightarrow \quad x = \frac{-1 \pm i\sqrt{7}}{2}$$

$\therefore x = 1$ is the rational root.

(ii) $x^8 - 3x + 1 = 0$

Here $a_n = 1$, $a_0 = 1$

If $\frac{p}{q}$ is a root of the polynomial, then as $(p, q) = 1$ p is a factor of $a_0 = 1$ and q is a factor of $a_n = 1$

Since 1 has no factors, the given equation has no rational roots.

3. Solve : $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63.$

Sol. $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63 \Rightarrow 8\left[\left(x^{\frac{1}{2n}}\right)^3 - \left(x^{\frac{-1}{2n}}\right)^3\right] = 63$

put $x^{\frac{1}{2n}} = y$

$$\Rightarrow \quad 8\left(y^3 - \frac{1}{y^3}\right) = 63$$

$$\Rightarrow \quad y^3 - \frac{1}{y^3} = \frac{63}{8} \Rightarrow \frac{y^6 - 1}{y^3} = \frac{63}{8}$$

$$\Rightarrow \quad 8y^6 - 8 = 63y^3$$

$$\Rightarrow \quad 8y^6 - 63y^3 - 8 = 0$$

$$\Rightarrow \quad 8t^2 - 63t - 8 = 0 \quad [\text{where } t = y^3]$$

$$\Rightarrow \quad (8t - 1)(t - 8) = 0$$

$$\Rightarrow \quad t = \frac{1}{8}, 8$$

Case (i) when $t = 8$, $\Rightarrow y^3 = 8 \Rightarrow y^3 = 2^3$

$$\Rightarrow \quad y = 2$$

Case (ii) when $t = \frac{1}{8}$, $y^3 = \frac{1}{8}$

$$\Rightarrow \quad y = \frac{1}{2}$$

When $y = 2$, $x^{\frac{1}{2n}} = 2$

$$\Rightarrow \quad x = (2)^{2n}$$

$$\Rightarrow \quad x = (2^2)^n$$

$$\Rightarrow \quad x = 4^n$$

When $y = \frac{1}{2}$, $x^{\frac{1}{2n}} = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^{2n}$

$$\Rightarrow \quad x = \left(\frac{1}{2^2}\right)^n = \frac{1}{4^n}$$

Hence the roots are 4^n.

4. Solve : $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}.$

Sol. Put $\sqrt{\frac{x}{a}} = y \Rightarrow 2y + \frac{3}{y} = \frac{b}{a} + \frac{6a}{b}$

$$\Rightarrow \quad \frac{2y^2 + 3}{y} = \frac{b^2 + 6a^2}{ab}$$

$$\Rightarrow \quad ab(2y^2 + 3) = (b^2 + 6a^2)y$$

$$\Rightarrow 2ab\,y^2 + 3ab - y(b^2 + 6a^2) = 0$$

$$\Rightarrow \quad 2aby^2 - y(b^2 + 6a^2) + 3ab = 0$$

$$\Rightarrow \quad 2ab\,y^2 - b^2y - 6a^2y + 3ab = 0$$

$$\Rightarrow \quad by(2ay - b) - 3a\,(2ay - b) = 0$$

$$\Rightarrow \quad (2ay - b)\,(by - 3a) = 0$$

$$\Rightarrow \quad 2ay = b,\ by = 3a$$

$$\Rightarrow \quad y = \frac{b}{2a},\ y = \frac{3a}{b}$$

Case (i) When $y = \frac{b}{2a}$

$$\Rightarrow \sqrt{\frac{x}{a}} = \frac{b}{2a} \Rightarrow \frac{x}{a} = \frac{b^2}{4a^2} \Rightarrow x = \frac{b^2}{4a}$$

Case (ii) When $y = \frac{3a}{b}$

$$\sqrt{\frac{x}{a}} = \frac{3a}{b} \Rightarrow \frac{x}{a} = \frac{9a^2}{b^2} \Rightarrow x = \frac{9a^3}{b^2}$$

$\therefore$The roots are $\frac{b^2}{4a}, \frac{9a^3}{b^2}$

CHAPTER
5

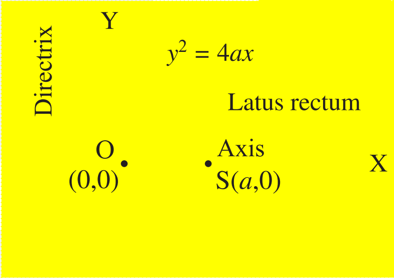
TWO DIMENSIONAL ANALYTICAL GEOMETRY - II

MUST KNOW DEFINITIONS

- A circle is the locus of a point in a plane which moves such that its distance from a fixed point is always constant.
- Tangent of a circle is a line which touches the circle at only one point.
- Normal is a line perpendicular to the tangent.
- From any point outside the circle two tangents can be drawn.
- **Parabola**

S(a, 0) → Focus O(0, 0) → Vertex
Y = 0 → Axis
Length of latus rectum = 4a

Parabola



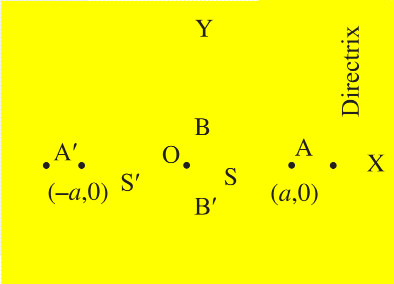
- **Ellipse**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

S(ae, 0), S′($-ae$, 0) → Foci
A(a, 0), A′($-a$, 0) → Vertices
O (0, 0) Centre
AA′ → Major axis → $2a$; BB′ → Minor axis → $2b$
$b^2 = a^2(1-e^2)$

Directrices $x=\frac{a}{e}$, $x=\frac{-a}{e}$

Ellipse



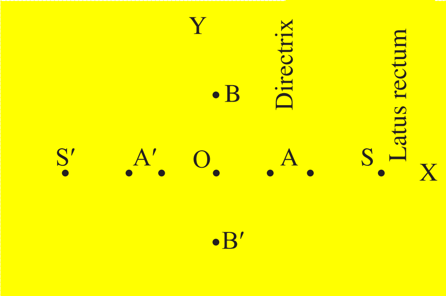
- Sum of the focal distances of any point on the ellipse is equal to length of the major axis.
- **Hyperbola**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

AA′ = $2a$ → Transverse axis; BB′ = $2b$ → Conjuate axis
O(0, 0) → centre
A(a, 0), A′($-a$, 0) → vertices; S(ae, 0), S′($-ae$, 0) → Focus

Directrices $x=\pm\frac{a}{e}$; Latus rectum = $\frac{2b^2}{a}$; $b^2 = a^2(e^2-1)$

Hyperbola

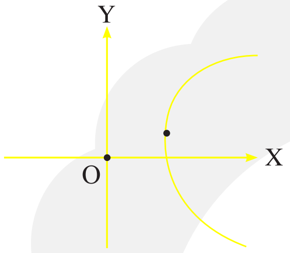
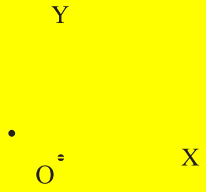
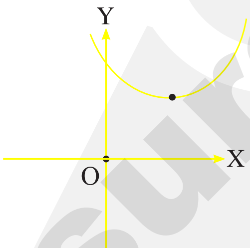
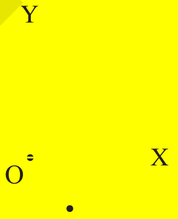


IMPORTANT FORMULA TO REMEMBER

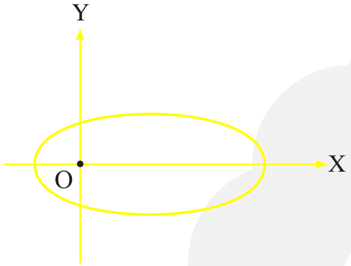
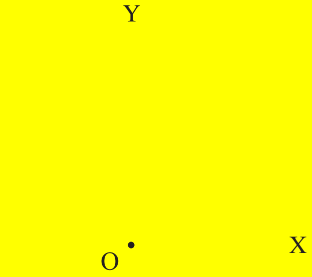
- Equation of circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.
- Equation of circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = 1^2$.
- For the equation $x^2 + y^2 + 2gx + 2fy + c = 0$, ... (1) centre is $(-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$
- (1) represents a real circle if $g^2 + f^2 - \text{c} > 0$, a point circle if $g^2 + f^2 - c = 0$ and an imaginary circle if $g^2 + f^2 - c < 0$.
- Equation of circle with (x_1, y_1) and (x_2, y_2) as extremities of diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
- If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is $\begin{cases} > 0, & (x_1, y_1) \text{ lies outside} \\ = 0, & (x_1, y_1) \text{ lies on the circle} \\ < 0, & (x_1, y_1) \text{ lies inside} \end{cases}$
- Equation of tangent at (x_1, y_1) to the circle is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- Equation of tangent to the circle is $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$
- Equation of normal to $x^2 + y^2 = a^2$ is $xy_1 - yx_1 = a^2$.
- Condition for $\text{y} = \text{mx} + \text{c}$ to be a tangent to the circle is $c^2 = a^2(1 + m^2)$

  Point of contact is $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$ or $\left(\frac{am}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}\right)$
- If $e = 1$, then the conic is a parabola, $e < 1$, then it is a ellipse and $e > 1$, it is a hyperbola.

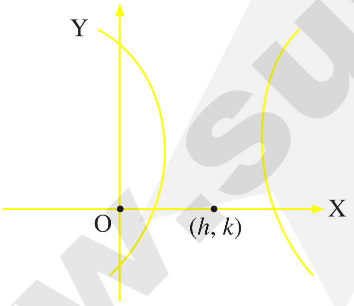
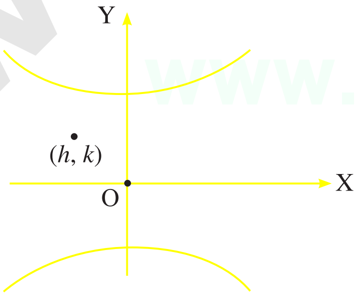
Parabola

Equation	Graph	Vertex	Focus	Axis of symmetry	Equation of Directrix	Length of Latus rectum
$(y-k)^2 = 4a(x-h)$		(h, k)	$(h+a,$ $0+k)$	$y = k$	$x = h - a$	$4a$
$(y-k)^2 = -4a(x-h)$		(h, k)	$(h-a,$ $0+k)$	$y = k$	$x = h + a$	$4a$
$(x-h)^2 = 4a(y-k)$		(h, k)	$(0+h,$ $a+k)$	$x = h$	$y = k - a$	$4a$
$(x-h)^2 = -4a(y-k)$		(h, k)	$(0+h,$ $-a+k)$	$x = h$	$y = k + a$	$4a$

Ellipse

Equation & Diagram	Centre	Major axis	Vertices	Foci
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a^2>b^2$ 	(h, k)	parallel to the x-axis	$(h+a, k)$	$(h+c, k)$
$(y-k)^2 = -4a(x-h)$ 	(h, k)	parallel to the y-axis	$(h, k+a)$	$(h, k+c)$

Hyperbola

Hyperbola	Transverse axis parallel to the x-axis
	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ Centre $\rightarrow (h, k)$ Vertices $(h+a, k)\ (h-a, k)$ Foci $\rightarrow (h+c, k)(h-c, k)$ Equation of directrix $= \pm\frac{a}{e}$
	**Transverse axis parallel to the y-axis** $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ Centre $\rightarrow (h, k)$ Vertices $(h, k+a)\ (h, k-a)$ Foci $(h, k+c)(h, k-c)$ Equation of directrices $y = \pm\frac{a}{e}$

✦ Parametric equations

Conic	Parametric equations	Parameter	Range of parameter	Any point on the conic
Circle	$x = a \cos \theta$, $y = a \sin \theta$	θ	$0 \le \theta \le 2\pi$	θ or $(a \cos \theta, a \sin \theta)$
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	t or $(at^2, 2at)$
Ellipse	$x = a \cos \theta$, $y = b \sin \theta$	θ	$0 \le \theta \le 2\pi$	θ or $(a \cos \theta, b \sin \theta)$
Hyperbola	$x = a \sec \theta$, $y = b \tan \theta$	θ	$-\pi \le \theta \le \pi$ except $\theta = \pm\dfrac{\pi}{2}$	θ or $(a \sec \theta, b \tan \theta)$

Parabola

- Equation of tangent is $yy_1 = 2a(x + x_1)$ (cartesian form) and in parametric form is $yt = x + at^2$.
- Equation of normal is $xy_1 + 2ay = x_1y_1 + 2ay_1$ in parametric form is $y + xt = at^3 + 2at$.

Ellipse

- Equation of tangent is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ (cartesian form)

  $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ (Parametric form)

- Equation of normal is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$ (cartesian form)

  $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$ (parametric form)

Hyperbola

- Equation of tangent is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ (cartesian form)

  $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ (parametric form)

- Equation of normal is $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$ (cartesian form)

  $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$ (parametric form)

- Condition for $y = mx + c$ to be a tangent to

  (i) Parabola $y^2 = 4ax$ is $c = \dfrac{a}{m}$; Point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

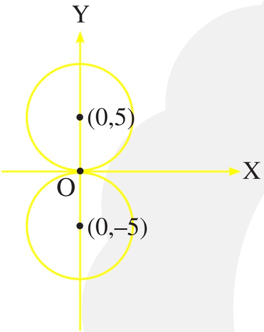
  (ii) Ellipse is $c^2 = a^2m^2 - b^2$; Point of contact is $\left(\dfrac{-a^2m}{c}, \dfrac{b^2}{c}\right)$

  (iii) Hyperbola is $c^2 = a^2m^2 - b^2$; Point of contact is $\left(\dfrac{-a^2m}{c}, \dfrac{-b^2}{c}\right)$

EXERCISE 5.1

1. Obtain the equation of the circles with radius 5 cm and touching x-axis at the origin in general form.

Sol. Given $r = 5$ cm



Since the circle touches the x axis, its centre is $(0, \pm 5)$

Equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

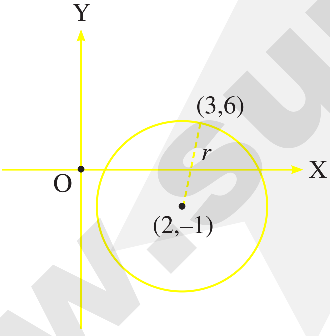
$$\Rightarrow \quad (x-0)^2 + (y \pm 5)^2 = 5^2$$

$$\Rightarrow \quad x^2 + y^2 + \cancel{25} \pm 10y = \cancel{25}$$

$$\Rightarrow \quad x^2 + y^2 \pm 10y = 0$$

2. Find the equation of the circle with centre (2, –1) and passing through the point (3, 6) in standard form.

Sol.



Given centre is $(2, -1)$ and passing through the point $(3, 6)$

$\therefore$ r = distance between $(2, -1)$ and $(3, 6)$

$$= \sqrt{(2-3)^2 + (-1-6)^2}$$

$$= \sqrt{(-1)^2 + (-7)^2}$$

$$= \sqrt{1+49} = \sqrt{50}$$

$\therefore$ Equation of the circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-2)^2 + (y+1)^2 = (\sqrt{50})^2$$

$$\Rightarrow \quad (x-2)^2 + (y+1)^2 = 50$$

3. Find the equation of circles that touch both the axes and pass through (– 4, – 2) in general form.

Sol. Since the circle touch both the axis. Its equation will be $(x-a)^2 + (y-a)^2 = a^2$ ... (1)

It passes through $(-4, -2)$

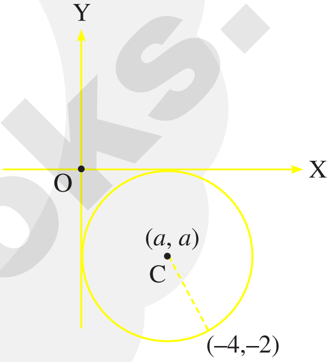
$$\therefore (-4-a)^2 + (-2-a)^2 = a^2$$

$$16 + \cancel{a^2} + 8a + 4 + a^2 + 4a = \cancel{a^2}$$

$$\Rightarrow \quad a^2 + 12a + 20 = 0$$

$$\Rightarrow \quad (a+10)(a+2) = 0$$

$$a = -10 \text{ or } -2$$



Case (i)

When $a = -10$, (1) becomes

$$(x+10)^2 + (y+10)^2 = -10^2$$

$$\Rightarrow \quad x^2 + 100 + 20x + y^2 + \cancel{100} + 20y = \cancel{100}$$

$$\Rightarrow \quad x^2 + y^2 + 20x + 20y + 100 = 0$$

Case (ii)

When $a = -2$, (1) becomes

$$(x+2)^2 + (y+2)^2 = -2^2$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 + 4y + \cancel{4} = \cancel{4}$$

$$\Rightarrow \quad x^2 + y^2 + 4x + 4y + 4 = 0$$

Hence, equation of the circles are

$$x^2 + y^2 + 4x + 4y + 4 = 0$$

or $$x^2 + y^2 + 20x + 20y + 100 = 0$$

4. Find the equation of the circle with centre (2, 3) and passing through the intersection of the lines 3x – 2y – 1 = 0 and 4x + y – 27 = 0.

Sol. Given centre is $(2, 3)$

Let us solve $3x - 2y = 1$ ...(1)

and $4x + y = 27$ ...(2)

$(1) \rightarrow \quad 3x - 2\cancel{y} = 1$

$(2) \times 2 \rightarrow \quad 8x + \cancel{2y} = 54$

$11x = 55 \Rightarrow x = 5$

Substituting $x = 5$ in equation (1)

$$\begin{aligned} \therefore\ 3(5) - 2y &= 1 \\ \Rightarrow \quad 15 - 2y &= 1 \\ \Rightarrow \quad 15 - 1 &= 2y \\ \Rightarrow \quad 14 &= 2y \\ \Rightarrow \quad y &= 7 \end{aligned}$$

The circle passes through (5, 7)

[∵ distance between (5, 7) and (2, 3)]

$$\begin{aligned} r &= \sqrt{(5-2)^2 + (7-3)^2} \\ &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9+16} = \sqrt{25} = 5 \end{aligned}$$

Equation of the circle is

$$\begin{aligned} (x - h)^2 + (y - k)^2 &= r^2 \\ \Rightarrow \quad (x - 2)^2 + (y - 3)^2 &= 5^2 \\ \Rightarrow \quad x^2 - 4x + 4 + y^2 - 6y + 9 &= 25 \\ \Rightarrow \quad x^2 + y^2 - 4x - 6y + 13 - 25 &= 0 \\ \Rightarrow \quad x^2 + y^2 - 4x - 6y - 12 &= 0 \end{aligned}$$

5. Obtain the equation of the circle for which (3, 4) and (2, –7) are the ends of a diameter.

Sol. Given ends of diameter are (3, 4),(2, –7) **[PTA -3]**

∴ Equation of the circle is

$$\begin{aligned} (x - x_1)(x - x_2) + (y - y_1)(y - y_2) &= 0 \\ \Rightarrow \quad (x - 3)(x - 2) + (y - 4)(y + 7) &= 0 \\ \Rightarrow \quad x^2 - 2x - 3x + 6 + y^2 + 7y - 4y - 28 &= 0 \\ \Rightarrow \quad x^2 + y^2 - 5x + 3y - 22 &= 0 \end{aligned}$$

6. Find the equation of the circle through the points (1, 0), (–1, 0) and (0, 1).

Sol. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

It passes through (1, 0)

$$\begin{aligned} \Rightarrow \quad 1^2 + 0 + 2g(1) + 2f(0) + c &= 0 \\ \Rightarrow \quad 2g + c &= -1 \quad \text{...(2)} \end{aligned}$$

It passes through (–1, 0)

$$\begin{aligned} \Rightarrow \quad (-1)^2 + 0 + 2g(-1) + 2f(0) + c &= 0 \\ \Rightarrow \quad -2g + c &= -1 \quad \text{...(3)} \end{aligned}$$

Also it passes through (0, 1)

$$\begin{aligned} \Rightarrow \quad 0 + 1^2 + 2g(0) + 2f(1) + c &= 0 \\ \Rightarrow \quad 2f + c &= -1 \quad \text{...(4)} \\ (2) + (3) \Rightarrow \quad 2c &= -2 \\ \Rightarrow \quad c &= -1 \end{aligned}$$

Substituting $c = -1$ in (2), we get

$$\begin{aligned} 2g - 1 &= -1 \\ \Rightarrow \quad 2g &= 0 \\ \Rightarrow \quad g &= 0 \end{aligned}$$

Substituting $c = -1$ in (4) we get,

$$\begin{aligned} 2f - 1 &= -1 \\ \Rightarrow \quad 2f &= 0 \\ \Rightarrow \quad f &= 0 \end{aligned}$$

∴ (1) becomes,

$$\begin{aligned} x^2 + y^2 + 0 + 0 - 1 &= 0 \\ \Rightarrow \quad x^2 + y^2 &= 1 \end{aligned}$$

7. A circle of area 9π square units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$. Find the equation of the circle.

Sol. Area of the circle = 9π sq. units **[Sep. - 2020]**

$\pi r^2 = 9\pi \Rightarrow r^2 = 9 \Rightarrow r = 3$

Diameters are $x + y = 5$ (1) and $x - y = 1$ (2)

We know that centre is the point of intersection of diameters.

∴ To find the centre, solve (1) and (2).

$$\begin{aligned} (1) + (2) \Rightarrow \quad x + \cancel{y} &= 5 \\ x - \cancel{y} &= 1 \\ \hline 2x &= 6 \\ \Rightarrow \quad x &= 3 \end{aligned}$$

Substituting $x = 3$ in $x + y = 5$

$$\begin{aligned} \therefore (1) \Rightarrow \quad 3 + y &= 5 \\ \Rightarrow \quad y &= 5 - 3 = 2 \end{aligned}$$

∴ Centre is (3, 2)

Hence, equation of the circle is

$$\begin{aligned} (x - h)^2 + (y - k)^2 &= r^2 \\ \Rightarrow \quad (x - 3)^2 + (y - 2)^2 &= 3^2 \\ \Rightarrow \quad x^2 - 6x + \cancel{9} + y^2 - 4y + 4 &= \cancel{9} \\ \Rightarrow \quad x^2 + y^2 - 6x - 4y + 4 &= 0 \end{aligned}$$

8. If $y = 2\sqrt{2}x + c$ is a tangent to the circle $x^2 + y^2 = 16$, find the value of c.

Sol. Given that equation of the circle is

$$\begin{aligned} x^2 + y^2 &= 16 \\ \Rightarrow \quad a^2 &= 16 \end{aligned}$$

and equation of the tangent is

$$y = 2\sqrt{2}x + c$$

$\Rightarrow \ m = 2\sqrt{2}$ and $c = c$ [∵ $y = mx + c$ is the tangent]

The condition for the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ is

$$\begin{aligned} c^2 &= a^2(1 + m^2) \\ \Rightarrow \quad c^2 &= 16(1 + (2\sqrt{2})^2) \\ \Rightarrow \quad c^2 &= 16(1 + 8) \\ \Rightarrow \quad c^2 &= 16(9) \\ \Rightarrow \quad c &= \pm 4(3) \\ \Rightarrow \quad c &= \pm 12 \end{aligned}$$

9. Find the equation of the tangent and normal to the circle $x^2 + y^2 - 6x + 6y - 8 = 0$ at (2, 2).

Sol. Equation of the circle is $x^2 + y^2 - 6x + 6y - 8 = 0$.

$\therefore$ Equation of the tangent at (x_1, y_1) is

$$xx_1 + yy_1 - \frac{6}{2}(x + x_1) + \frac{6}{2}(y + y_1) - 8 = 0$$

Given (x_1, y_1) is (2, 2)

Equation of the tangent at (2, 2) is

$$x(2) + y(2) - 3(x + 2) + 3(y + 2) - 8 = 0$$

$$\Rightarrow \quad 2x + 2y - 3x - \cancel{6} + 3y + 6 - 8 = 0$$

$$\Rightarrow \quad -x + 5y - 8 = 0$$

$$\Rightarrow \quad x - 5y + 8 = 0$$

Equation of the normal is

$$yx_1 - xy_1 + g(y - y_1) - f(x - x_1) = 0$$

$$\Rightarrow y(2) - x(2) - 3(y - 2) - 3(x - 2) = 0$$

$$[\because 2g = -6 \Rightarrow g = -3;\ 2f = 6 \ \Rightarrow f = 3]$$

$$\Rightarrow \quad 2y - 2x - 3y + 6 - 3x + 6 = 0$$

$$\Rightarrow \quad -5x - y + 12 = 0$$

$$\Rightarrow \quad 5x + y - 12 = 0$$

10. Determine whether the points (–2, 1), (0, 0) and (–4, –3) lie outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$.

Sol. (i) Given equation of the circle is

$$x^2 + y^2 - 5x + 2y - 5 = 0$$

At (–2, 1), (1) becomes

$$(-2)^2 + 1^2 - 5(-2) + 2(1) - 5$$
$$= 4 + 1 + 10 + 2 - 5$$
$$= 17 - 5 = 12 > 0.$$

$\therefore$ (–2, 1) lies outside the circle.

(ii) At (0, 0), (1) becomes –5 < 0

$\therefore$ (0, 0) lies inside the circle.

(iii) At (–4, –3), (1) becomes

$$(-4)^2 + (-3)^2 - 5(-4) + 2(-3) - 5$$
$$= 16 + 9 + 20 - 6 - 5$$
$$= 45 - 11 = 34 > 0$$

$\therefore$ (–4, –3) lies outside the circle.

11. Find centre and radius of the following circles.

(i) $x^2 + (y + 2)^2 = 0$

(ii) $x^2 + y^2 + 6x - 4y + 4 = 0$

(iii) $x^2 + y^2 - x + 2y - 3 = 0$

(iv) $2x^2 + 2y^2 - 6x + 4y + 2 = 0$ **[PTA-6]**

Sol. (i) Equation of the circle is $x^2 + (y + 2)^2 = 0$

Centre is (0, –2) and radius is 0.

(ii) Equation of the circle is

$x^2 + y^2 + 6x - 4y + 4 = 0$.

Here $2g = 6 \ \Rightarrow g = 3$

$2f = -4 \Rightarrow f = -2$ and $c = 4$

Centre is $(-g, -f) \Rightarrow (-3, 2)$

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + (-2)^2 - 4}$$
$$= \sqrt{9 + 4 - 4} = \sqrt{9}$$
$$= 3 \text{ units}$$

(iii) Equation of the circle is $x^2 + y^2 - x + 2y - 3 = 0$

$$\text{Here } 2g = -1 \Rightarrow g = \frac{-1}{2}$$

$$2f = 2 \ \Rightarrow f = 1 \text{ and } c = -3$$

$$\text{Centre is } (-g, -f) = \left(\frac{1}{2}, -1\right)$$

$$\text{and } r = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{\frac{1}{4} + 1 + 3}$$
$$= \sqrt{\frac{1}{4} + 4} = \sqrt{\frac{1 + 16}{4}}$$
$$r = \frac{\sqrt{17}}{2} \text{ units.}$$

(iv) Equation of the circle is

$$2x^2 + 2y^2 - 6x + 4y + 2 = 0$$

Dividing by 2, we get

$$x^2 + y^2 - 3x + 2y + 1 = 0$$

$$\text{Here } 2g = -3 \Rightarrow g = \frac{-3}{2}$$

$$2f = 2 \ \Rightarrow f = 1$$

$$\text{and } c = 1$$

$$\therefore \text{Centre is } (-g, -f) = \left(\frac{3}{2}, -1\right)$$

$$\text{and } r = \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{3}{2}\right)^2 + 1^2 - 1}$$
$$= \sqrt{\frac{9}{4}} = \frac{3}{2} \text{ units.}$$

12. If the equation $3x^2 + (3 - p)\,xy + qy^2 - 2px = 8pq$ represents a circle, find p and q. Also determine the centre and radius of the circle.

Sol. Given equation of the circle is **[PTA - 1]**

$$3x^2 + (3 - p)xy + qy^2 - 2px = 8pq$$

For the circle, co-efficient of $xy = 0$

$$\Rightarrow \quad 3 - p = 0 \Rightarrow p = 3$$

Also, co-efficient of x^2 = co-efficient of y^2

$$\Rightarrow \quad 3 = q$$

∴ Equation of the circle is

$$\begin{aligned} 3x^2 + 3y^2 - 6x &= 8(3)(3) \\ 3x^2 + 3y^2 - 6x - 72 &= 0 \end{aligned}$$

Dividing by 3, we get

$$\begin{aligned} x^2 + y^2 - 2x - 24 &= 0 \\ \text{Here } 2g &= -2 \\ \Rightarrow \quad g &= -1 \\ f &= 0 \text{ and } c = -24 \\ \text{Centre is } (-g, -f) &= (1, 0) \\ \text{and } r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{(-1)^2 + 0 + 24} \\ &= \sqrt{25} \\ &= 5 \text{ units.} \end{aligned}$$

EXERCISE 5.2

1. Find the equation of the parabola in each of the cases given below :

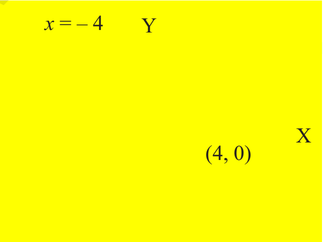
(i) focus (4, 0) and directrix $x = -4$. **[Aug. - 2021]**

(ii) passes through (2, –3) and symmetric about y- axis.

(iii) vertex(1, –2) and focus(4, –2).

(iv) end points of latus rectum (4, –8) and (4, 8).

Sol. (i) Given Focus (4, 0) and directrix $x = -4$. Since (4, 0) is the focus and the directrix is $x = -4$, the parabola is of the form $y^2 = 4ax$.

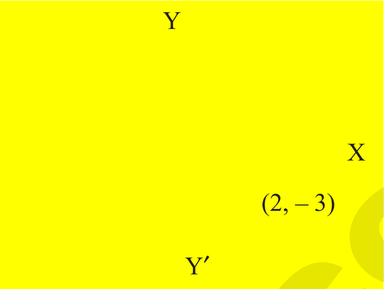


$$\text{Also } a = 4$$

∴ Equation of the parabola is $y^2 = 4(4)x$

$$\Rightarrow \quad y^2 = 16x$$

(ii) The parabola passes through (2, –3) and symmetric about y-axis.



Since the parabola passes through (2, –3) the parabola is open downward.

∴ Its equation is $x^2 = -4ay$ ...(1)

Substitute the point (2, –3) we get,

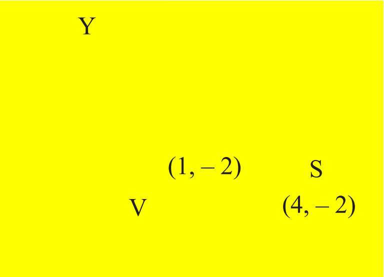
$$\begin{aligned} 2^2 &= -4a(-3) \\ \Rightarrow \quad 4 &= 12a \\ \Rightarrow \quad a &= \frac{\cancel{4}}{\cancel{12}_{3}} = \frac{1}{3} \end{aligned}$$

(1) becomes,

$$\begin{aligned} \therefore x^2 &= -4\left(\frac{1}{3}\right)y \\ \Rightarrow \quad 3x^2 &= -4y \end{aligned}$$

(iii) Vertex is (1, –2) and focus is (4, –2).

The parabola is right open.



a = distance between (1, –2) and (4, –2)

$$\begin{aligned} \Rightarrow \quad a &= \sqrt{(1-4)^2 + (-2+2)^2} \\ \Rightarrow \quad a &= \sqrt{(-3)^2} = 3 \end{aligned}$$

Equation of the parabola is

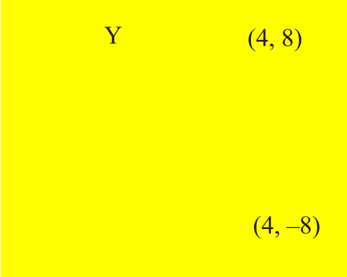
$$\begin{aligned} (y - k)^2 &= 4a\,(x - h) \\ (y + 2)^2 &= 4(3)\,(x - 1) \end{aligned}$$

[∵ (h, k) is $(1, -2)$]

$$\Rightarrow \quad (y + 2)^2 = 12\,(x - 1)$$

(iv) End points of latus rectum are (4, –8) and (4, 8).

Focus is the mid-point of (4, –8) and (4, 8)



$$\therefore \text{Focus} = \left(\frac{4+4}{2}, \frac{-8+8}{2}\right) = (4, 0)$$

$$\therefore a = 4 \text{ and vertex is } (0, 0)$$

Equation of the parabola is $y^2 = 4ax$

$$\Rightarrow \quad y^2 = 4(4)x$$

$$\Rightarrow \quad y^2 = 16x$$

2. Find the equation of the ellipse in each of the cases given below :

(i) foci (± 3, 0), $e = \frac{1}{2}$.

(ii) foci (0, ± 4) and end points of major axis are (0, ± 5).

(iii) length of latus rectum 8, eccentricity = $\frac{3}{5}$ and major axis on x-axis.

(iv) length of latus rectum 4, distance between foci $4\sqrt{2}$, centre (0, 0) and major axis as y - axis.

Sol. (i) foci (±3, 0), $e = \frac{1}{2}$

Since foci are $(\pm ae, 0)$

$$\Rightarrow \quad ae = 3$$

$$\Rightarrow \quad a \cdot \frac{1}{2} = 3 \Rightarrow a = 6$$

Centre is (0, 0)

$$\text{and } b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad b^2 = 36\left(1 - \frac{1}{4}\right)$$

$$\Rightarrow \quad b^2 = 36\left(\frac{3}{4}\right)$$

$$\Rightarrow \quad b^2 = 27$$

$\therefore$ Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{x^2}{36} + \frac{y^2}{27} = 1$$

(ii) Foci (0, ±4) and end points of major axis are (0, ±5)

Since foci are $(0, \pm be) \Rightarrow be = 4$

End points of major axis are (0, ±5)

$$\Rightarrow \quad b = 5$$

$$\therefore 5(e) = 4 \Rightarrow \quad e = \frac{4}{5}$$

$$\text{Also, } a^2 = b^2(1 - e^2)$$

$$\Rightarrow \quad a^2 = 25\left(1 - \frac{16}{25}\right) = 25\left(\frac{25-16}{25}\right)$$

$$\Rightarrow \quad a^2 = 9$$

Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{25} = 1$$

(iii) Length of latus rectrum = 8, $e = \frac{3}{5}$ and major axis on x-axis.

$$\text{Given } \frac{2b^2}{a} = 8, e = \frac{3}{5}$$

$$b^2 = 4a$$

$$b^2 = a^2(1 - e^2)$$

$$4a = a^2\left(1 - \frac{9}{25}\right)$$

$$4 = a\left(\frac{25-9}{25}\right)$$

$$100 = a(16)$$

$$a = \frac{100}{16} = \frac{25}{4} \Rightarrow a^2 = \frac{625}{16}$$

$$b^2 = 4 \times \frac{25}{4} = 25$$

Since major axis is on x-axis, equation of the ellipse is

$$\frac{(x-0)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{\frac{625}{16}} + \frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{16x^2}{625} + \frac{y^2}{25} = 1$$

(iv) Length of latus rectum = 4, distance between foci = $4\sqrt{2}$ major axis is y-axis.

Given $\frac{2b^2}{a} = 4$ and distance between foci $= 2ae = 4\sqrt{2}$

$$\Rightarrow \quad ae = 2\sqrt{2}$$

$$\Rightarrow \quad a^2e^2 = 8 \quad \text{...(1)}$$

$$\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a \qquad ...(2)$$

We know $b^2 = a^2(1 - e^2)$

$\Rightarrow \quad b^2 = a^2 - a^2e^2$

$\Rightarrow \quad 2a = a^2 - 8$

[using (1) and (2)]

$\Rightarrow a^2 - 2a - 8 = 0$

On factorising we get

$(a - 4)(a + 2) = 0$

$\Rightarrow \quad a = 4$ or -2

$a = 4$

[$\because a = -2$ is not possible]

$\Rightarrow \quad a^2 = 16$

$\therefore$ From (2), $b^2 = 2(4) = 8$

Hence, the equation of the ellipse is

$$\frac{x^2}{8} + \frac{y^2}{16} = 1 \qquad [\because \text{ Major axis is } y\text{-axis}]$$

3. Find the equation of the hyperbola in each of the cases given below:

(i) foci (±2, 0) eccentricity = $\frac{3}{2}$

(ii) Centre (2,1), one of the foci (8,1) and corresponding directrix $x = 4$.

(iii) passing through (5, –2)and length of the transverse axis along x axis and of length 8 units.

Sol. (i) Given foci (±2, 0) ⇒ $ae = 2$ and centre is (0, 0)

$$a\left(\frac{3}{2}\right) = 2 \Rightarrow a = \frac{4}{3}$$

$$b^2 = a^2(e^2 - 1)$$

$$\Rightarrow b^2 = \frac{16}{9}\left(\frac{9}{4} - 1\right) = \frac{16}{9}\left(\frac{9-4}{4}\right) = \frac{\overset{4}{\cancel{16}}}{9} \times \frac{5}{\cancel{4}}$$

$$\Rightarrow \quad b^2 = \frac{4 \times 5}{9} = \frac{20}{9}$$

$\therefore$ Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\Rightarrow \qquad \frac{x^2}{\frac{16}{9}} - \frac{y^2}{\frac{20}{9}} = 1$$

$$\Rightarrow \qquad \frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

(ii) ae = distance between centre and focus

$$ae = \sqrt{(8-2)^2 + (1-1)^2} = \sqrt{6^2} = 6 \qquad ...(1)$$

Also $\frac{a}{e}$ = distance between centre and directrix

$$\frac{a}{e} = \sqrt{(4-2)^2 + (1-1)^2} = \sqrt{2^2} = 2$$

[$\because$ (4, 1) is a point on the directrix]

$$(1) \times (2) \rightarrow a\cancel{e} \times \frac{a}{\cancel{e}} = 6 \times 2$$

$\Rightarrow \quad a^2 = 12$

$(1) \rightarrow \quad a^2e^2 = 36$

$12(e^2) = 36$

$\Rightarrow \quad e^2 = 3$

$\Rightarrow \quad e = \sqrt{3}$

Also, $b^2 = a^2(e^2 - 1) = 12(3 - 1) = 12(2) = 24$

$\therefore$ Equation of the hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$$

(iii) Passing through (5, –2) length of the transverse axis is a long x-axis and of length 8 units.

$$2a = 8 \Rightarrow a = 4$$

Since the transverse axis is along x-axis, centre is (0, 0)

Equation of the hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{b^2} = 1$$

Since (5, –2) passes through the parabola,

$$\frac{25}{16} - \frac{4}{b^2} = 1 \Rightarrow \frac{4}{b^2} = \frac{25}{16} - 1 = \frac{25-16}{16} = \frac{9}{16}$$

$$\therefore b^2 = \frac{16 \times 4}{9} = \frac{64}{9}$$

$\therefore$ Equation of the hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{\frac{64}{9}} = 1 \Rightarrow \frac{x^2}{16} - \frac{9y^2}{64} = 1$$

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

(i) $y^2 = 16x$ (ii) $x^2 = 24y$

(iii) $y^2 = -8x$ (iv) $x^2 - 2x + 8y + 17 = 0$

(v) $y^2 - 4y - 8x + 12 = 0$

Sol. (i) $y^2 = 16x$

The given parabola is right open parabola and $4a = 16 \Rightarrow a = 4$.

(a) Vertex is (0, 0)

$\Rightarrow h = 0, k = 0$

(b) focus is $(h + a, 0 + k)$

$\Rightarrow (0 + 4, 0 + 0) = (4, 0)$

(c) Equation of directrix is $x = h - a$

$\Rightarrow x = 0 - 4 \Rightarrow x = -4$

(d) Length of latus rectum is $4a = 16$.

(ii) $x^2 = 24y$

The given parabola is open upward parabola and $4a = 24 \Rightarrow a = 6$.

(a) Vertex is (0, 0)

$\Rightarrow h = 0, k = 0$

(b) focus is $(0 + h, a + k)$

$\Rightarrow (0 + 0, 6 + 0) = (0, 6)$

(c) Equation of directrix is $y = k - a$

$\Rightarrow y = 0 - 6 \Rightarrow y = -6$

(d) Length of latus rectum is $4a = 24$.

(iii) $y^2 = -8x$

The given parabola is left open parabola and $4a = 8 \Rightarrow a = 2$.

(a) Vertex is (0, 0)

$\Rightarrow h = 0, k = 0$

(b) focus is $(h - a, 0 + k)$

$\Rightarrow (0 - 2, 0 + 0)$

$\Rightarrow (-2, 0)$

(c) Equation of directrix is $x = h + a$

$\Rightarrow x = 0 + 2 \Rightarrow x = 2$

(d) Length of latus rectum is $4a = 8$.

(iv) $x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x = -8y - 17$

Adding 1 both sides, we get

$x^2 - 2x + 1 = -8y - 17 + 1$

$\Rightarrow (x - 1)^2 = -8y - 16 = -8(y + 2)$

$\Rightarrow (x - 1)^2 = -8(y + 2)$

This is a open downward parabola, latus rectum $4a = 8 \Rightarrow a = 2$.

(a) Vertex is (1, –2)

$\Rightarrow h = 1, k = -2$

(b) focus is $(0 + h, - a + k)$

$\Rightarrow (0 + 1, -2 - 2)$

$\Rightarrow (1, -4)$

(c) Equation of directrix is $y = k + a$

$\Rightarrow y = -2 + 2 \Rightarrow y = 0$

(d) Length of latus rectum is $4a = 8$ units.

(v) $y^2 - 4y - 8x + 12 = 0$

$y^2 - 4y = 8x - 12$

Adding 4 both sides, we get,

$y^2 - 4y + 4 = 8x - 12 + 4 = 8x - 8$

$\Rightarrow (y - 2)^2 = 8(x - 1)$

This is a right open parabola and latus rectum is $4a = 8 \Rightarrow a = 2$.

(a) Vertex is (1, 2) $\Rightarrow h = 1, k = 2$

(b) focus is $(h + a, 0 + k)$

$\Rightarrow (1 + 2, 0 + 2)$

$\Rightarrow (3, 2)$

(c) Equation of directrix is $x = h - a$

$\Rightarrow x = 1 - 2$

$\Rightarrow x = -1$

(d) Length of latus rectum is $4a = 8$ units.

5. **Identify the type of conic and find centre, foci, vertices and directrices of each of the following.**

(i) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **(ii)** $\dfrac{x^2}{3} + \dfrac{y^2}{10} = 1$

(iii) $\dfrac{x^2}{25} - \dfrac{y^2}{144} = 1$ **(iv)** $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

Sol. (i) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ is in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Here $a^2 = 25$ and $b^2 = 9$

We know $e^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{25 - 9}{25} = \dfrac{16}{25}$

$e = \dfrac{4}{5}$ and $a = 5$

Hence $ae = 5 \times \dfrac{4}{5}$

$ae = 4$ and $\dfrac{a}{e} = \dfrac{5}{4} = 5 \times \dfrac{5}{4}$

$\dfrac{a}{e} = \dfrac{25}{4}$

Major axis x axis.

Center = (0, 0)

Foci = $(\pm ae, 0) = (\pm 4, 0)$

Vertices = $(\pm a, 0) = (\pm 5, 0)$

Directrix $x = \pm \dfrac{a}{e} = \pm \dfrac{25}{4}$

(ii) Given equation is $\frac{x^2}{3}+\frac{y^2}{10}=1$ is in the form $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Here $a^2 = 10$ and $b^2 = 3$

We know $e^2 = \frac{a^2-b^2}{a^2} = \frac{10-3}{10} = \frac{7}{10}$

$e = \frac{\sqrt{7}}{\sqrt{10}}$ and $a = \sqrt{10}$

Hence $ae = \sqrt{10} \times \frac{\sqrt{7}}{\sqrt{10}}$

$ae = \sqrt{7}$ and

$\frac{a}{e} = \frac{\sqrt{10}}{\frac{\sqrt{7}}{\sqrt{10}}} = \sqrt{10} \times \frac{10}{\sqrt{7}}$

$\frac{a}{e} = \frac{10}{\sqrt{7}}$

Major axis y axis.

Center = $(0, 0)$

Foci = $(0, \pm ae) = (0, \pm\sqrt{7})$

Vertices = $(0, \pm a) = (0, \pm\sqrt{10})$

Directrix $y = \pm\frac{a}{e} = \pm\frac{10}{\sqrt{7}}$

(iii) Given equation is $\frac{x^2}{25}-\frac{y^2}{144}=1$ is in the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Here $a^2 = 25$ and $b^2 = 144$

We know $e^2 = \frac{a^2+b^2}{a^2} = \frac{25+144}{25} = \frac{169}{25}$

$e = \frac{13}{5}$ and $a = 5$

Hence $ae = 5 \times \frac{13}{5}$

$ae = 13$ and $\frac{a}{e} = \frac{5}{\frac{13}{5}} = 5 \times \frac{5}{13}$

$\frac{a}{e} = \frac{25}{13}$

Transverse axis x axis.

Center = $(0, 0)$

Foci = $(0, \pm ae, 0) = (\pm 13, 0)$

Vertices = $(\pm a, 0) = (\pm 5, 0)$

Directrix $y = \pm\frac{a}{e} = \pm\frac{25}{13}$

(iv) $\frac{y^2}{16}-\frac{x^2}{9}=1$ is in the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

Here $a^2 = 16$ and $b^2 = 9$

We know $e^2 = \frac{a^2+b^2}{a^2} = \frac{16+9}{16} = \frac{25}{16}$

$e = \frac{5}{4}$ and $a = 4$

Hence $ae = 4 \times \frac{5}{4}$

$ae = 5$ and $\frac{a}{e} = \frac{4}{\frac{5}{4}} = 4 \times \frac{4}{5}$

$\frac{a}{e} = \frac{16}{5}$

Transverse axis y axis.

Center = $(0, 0)$

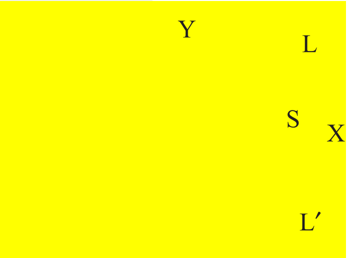
Foci = $(0, \pm ae, 0) = (0, \pm 5)$

Vertices = $(0, \pm a) = (0, \pm 4)$

Directrix $y = \pm\frac{a}{e} = \pm\frac{16}{5}$

6. **Prove that the length of the latus rectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$.** **[PTA - 2]**

Sol. The latusrectum LL′ of the hyperbola passes through S$(ae, 0)$



∴ L is (ae, y_1)

Substituting L in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ we get,

$$\frac{a^2e^2}{a^2}-\frac{y_1^2}{b^2}=1 \Rightarrow e^2-\frac{y_1^2}{b^2}=1$$

$$\Rightarrow \quad e^2-1=\frac{y_1^2}{b^2} \qquad [b^2 = a^2(e^2-1)$$

$$\Rightarrow \quad y_1^2 = b^2(e^2-1) \qquad \Rightarrow \frac{b^2}{a^2} = e^2-1]$$

$$\Rightarrow \quad y_1^2 = b^2\left(\frac{b^2}{a^2}\right) \qquad \Rightarrow \quad y_1^2 = \frac{b^4}{a^2}$$

12th STD

GOVT. SUPPLEMENTARY EXAM - AUG. 2021

Reg. No. ☐☐☐☐☐☐

Part - III

Mathematics (with answers)

Time Allowed : 3.00 Hours] [**Maximum Marks : 90**

Instructions :

(1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) **All** questions are **compulsory.**

(ii) Choose the most appropriate answer from the given **four** alternatives and write the option code and the corresponding answer.

20 × 1 = 20

1. In the set $\mathbb{R}$ of real numbers '*' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?

(1) $a*b = a$ (2) $a*b = \min(a.b)$ (3) $a*b = a^b$ (4) $a*b = \max(a,b)$

2. The value of $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right)$ is

(1) 0 (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) π

3. If P(x, y) be any point on $16x^2 + 25y^2 = 400$ with foci $F_1(3,0)$ and $F_2(-3,0)$ then $PF_1 + PF_2$ is

(1) 10 (2) 8 (3) 12 (4) 6

4. The volume of the parallelepiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}, \hat{i}+\hat{j}+\pi\hat{k}$ is

(1) π (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{3}$

5. $\int_0^{\frac{\pi}{2}} \sin^7 x \, dx =$

(1) $\frac{\pi}{2}$ (2) $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$ (3) 0 (4) 1

6. A pair of dice numbered 1, 2, 3, 4, 5, 6 of a six-sided die and 1, 2, 3, 4 of a four-sided die is rolled and the sum is determined. If the random variable X denote the sum, then the number of elements in the inverse image of 7 is

(1) 3 (2) 1 (3) 4 (4) 2

7. If A, B and C are invertible matrices of some order, then which one of the following is not true?

(1) $\det A^{-1} = (\det A)^{-1}$ (2) $\text{adj } A = |A|A^{-1}$

(3) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ (4) $\text{adj }(AB) = (\text{adj } A)(\text{adj } B)$

8. If the planes $\vec{r}.(2\hat{i}-\lambda\hat{j}+\hat{k}) = 3$ and $\vec{r}.(4\hat{i}+\hat{j}-\mu\hat{k}) = 5$ are parallel, then the value of λ and μ are respectively.

(1) $-\frac{1}{2}, -2$ (2) $\frac{1}{2}, -2$ (3) $\frac{1}{2}, 2$ (4) $-\frac{1}{2}, 2$

9. If we measure the side of a cube to be 4 cm with an error of 0.1 cm, then the error in our calculation of the volume is (in cubic cm) :

(1) 2 (2) 0.4 (3) 4.8 (4) 0.45

10. If the function $f(x) = \frac{1}{12}$ for $a < x < b$, represents a probability density function of a continuous random variable X, then which of the following cannot be the value of a and b?

(1) 7 and 19 (2) 0 and 12 (3) 16 and 24 (4) 5 and 17

11. The solution of $\frac{dy}{dx} + p(x)y = 0$ is

(1) $x = ce^{-\int pdy}$ (2) $y = ce^{\int pdx}$ (3) $x = ce^{\int pdy}$ (4) $y = ce^{-\int pdx}$

12. The position of a particle 's' moving at any time t is given by $s(t) = 5t^2 - 2t - 8$. The time at which the particle is at rest, is :

(1) 1 (2) 0 (3) 3 (4) $\frac{1}{3}$

13. The inverse of $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ is :

(1) $\begin{bmatrix} 3 & -1 \\ -5 & -3 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ (3) $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$ (4) $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$

14. The function $f(x) = x^2$, in the interval $[0, \infty)$ is :

(1) cannot be determined (2) increasing function

(3) increasing and decreasing function (4) decreasing function

15. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos x \, dx$ is :

(1) 0 (2) $\frac{3}{2}$ (3) $\frac{2}{3}$ (4) $\frac{1}{2}$

16. A zero of $x^3 + 64$ is :

(1) $4i$ (2) 0 (3) -4 (4) 4

17. The centre of the hyperbola $\frac{(x-1)^2}{16} - \frac{(y+1)^2}{25} = 1$ is :

(1) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ (2) $(-1, 1)$ (3) $(1, -1)$ (4) $(0, 0)$

18. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x^{\frac{1}{4}} = 0$ are :

(1) 2, 6 (2) 2, 3 (3) 2, 4 (4) 3, 3

19. If $|z| = 1$, then the value of $\frac{1+z}{1+\bar{z}}$ is :

(1) $\frac{1}{z}$ (2) z (3) 1 (4) $\bar{z}$

20. The value of the complex number $(i^{25})^3$ is equal to :

(1) 1 (2) i (3) $-i$ (4) -1

PART - II

Note : (i) Answer **any seven** questions.

(ii) Question number **30** is **compulsory.** **7 × 2 = 14**

21. If $z = (2 + 3i)(1 - i)$ then prove that $z^{-1} = \frac{5}{26} - i\frac{1}{26}$.

22. If α and β are the roots of $x^2 - 5x + 6 = 0$ then prove that $\alpha^2 - \beta^2 = \pm 5$.

23. For what value of x does $\sin x = \sin^{-1} x$?

24. Show that the three vectors $2\hat{i} + 3\hat{j} + \hat{k}$, $\hat{i} - 2\hat{j} + 2\hat{k}$ and $3\hat{i} + \hat{j} + 3\hat{k}$ are coplanar.

25. Prove that $\lim\limits_{x \to \infty} \left(\frac{e^x}{x^m}\right)$, where m is a positive integer, is ∞.

26. If $g(x) = x^2 + \sin x$, then find dg.

27. Show that the solution of $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ is $\sin^{-1} y = \sin^{-1} x + C$ (or) $\sin^{-1} x = \sin^{-1} y + C$.

28. If X is the random variable with distribution function F(x), given by : $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{2}(x^2 + x) & 0 \le x < 1 \\ 1, & 1 \le x < \infty \end{cases}$

then prove that the p.d.f. is $f(x) = \begin{cases} \frac{1}{2}(2x+1) & ;\ 0 \le x < 1 \\ 0 & ;\ \text{otherwise} \end{cases}$.

29. The probability density function of X is given by $f(x) = \begin{cases} kxe^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$ Prove that the value of k is 4.

30. Show that the differential equation corresponding to $y = A \sin x$, where A is an arbitrary constant, is $y = y' \tan x$.

PART - III

Note : (i) Answer **any seven** questions.

(ii) Question number **40** is **compulsory.** **7 × 3 = 21**

31. Show that the rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$ is 3.

32. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A|\, I$.

33. Show that the points 1, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ are the vertices of an equilateral triangle of side length $\sqrt{3}$.

34. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Show that the volume of the cuboid is 60 cubic units.