GOVERNMENT OF TAMIL NADU

HIGHER SECONDARY FIRST YEAR

PHYSICS

VOLUME - I

A publication under Free Textbook Programme of Government of Tamil Nadu

Department of School Education

Untouchability is Inhuman and a Crime

XI_Physics_First 8 pages Higher.indd 1 05-01-2022 22:21:57

Government of Tamil Nadu

First Edition - 2018
Revised Edition - 2019, 2020, 2022
(Published Under New Syllabus)

NOT FOR SALE

Content Creation

State Council of Educational
Research and Training

Printing & Publishing

Tamil NaduTextbook and Educational
Services Corporation

www.textbooksonline.tn.nic.in

CONTENTS

PHYSICS

5G7F8

E-book

6UAQJ

Assessment

HOW TO USE THE BOOK

Scope of Physics	• Awareness on higher learning -courses, institutions and required competitive exams • Financial assistance possible to help students to climb academic ladder
Learning Objectives:	• Overview of the unit • Gives clarity on the goals and intention of the topics
DO YOU KNOW?	• Additional facts related to the topics covered to facilitate curiosity driven learning
Example problems	• To ensure understanding, problems/illustrations are given at every stage before advancing to next level
ZNYP5	• Visual representation of concepts with illustrations • Videos, animations, and tutorials
ICT	• To harness the digital skills to class room learning and experimenting
Summary	• Recap of salient points of the lesson
Concept Map	• Schematic outline of salient learning of the unit
Evaluation	• Evaluate students' understanding and get them acquainted with the application of physical concepts to numerical and conceptual questions
Books for Reference	• List of relevant books for further reading
Solved examples	• Numerical/conceptual questions are solved to enable students to tackle standard problems in mechanics in appendix 1
Competitive Exam corner	• Model Questions - To motivate students aspiring to take up competitive examinations such as NEET, JEE, Physics Olympiad, JIPMER etc
Appendix	• Additional information including the chronological development of physics is provided
Glossary	• Scientific terms frequently used with their Tamil equivalents

Physics learning - Correct method

- The correct way to learn is to understand the concept, express the same in the language of mathematics which are equations.
- Each equation conveys the meaning of a phenomena or relationship between various parameters in the equations. Such relationship can be diagrammatically expressed as graphs.
- This interlink should be clear in mind while going through the entire text.

Value addition of the book

- Mathematical topics such as vectors, differentiation and integration are essential to understand and express physical phenomena.
- Inclusion of these topics and usage of vector notation wherever necessary is the salient feature of this book.
- Becoming familiar with vector notations and basic mathematics for physics will solve a lot of difficulty currently faced by students pursuing higher education in engineering, technology and science disciplines.

Scope of Physics - Higher Education

Exams

- JEE-Joint Entrance Examination
- Physics Olympiad Exam
- NEET- National Eligibility and Entrance Test
- NEST- National Entrance Screening Test
- AIEEE- All India Engineering Entrance Exam
- AIIMS- All India Institute of Medical Science (Entrance Examination)
- JIPMER- Jawaharlal institute of Postgraduate Medical Education and Research (Entrance Examination)
- KVPY- Kishore Vaigyanik Protsahan Yojana
- JAM- Joint Admission Test
- TIFR GS - Tata Institute of Fundamental Research Graduate School Admissions Examination
- JEST- Joint Entrance Screening Test
- NET- National Eligibility Test (CSIR and UGC)
- GATE-Graduate Aptitude Test in Engineering
- ICAR -AIEEA-Indian Council of Agricultural Research All India Entrance Examination

After completing +2

- B.Sc (Physics)
- Integrated M.Sc (Physics) (Central Universities)
- Integrated M.Sc (in Central Research Institutes through NEST and KVPY with stipend)
- B.Sc./B.S./B.Stat./B.Math./M.S. in Mathematics, Chemistry and Biology. (KVPY)
- B.E/B.Tech/ B.Arch (JEE, AIEEE in IITs and NITs) MBBS/ B.D.S/B.Pharm (NEET, JIPMER, AIIMS)
- B.Sc. (Agriculture) (ICAR -AIEEA)
- Dual Degree Program BS & MS (JEE, JEST in IITs and IISERs)
- B.Sc (Hospitality administration)
- B.Sc (Optoelectronics)
- B.Sc (Optometry)
- B.Tech (Optics and Optoelectronics)

After completing undergraduate course in Physics (B.Sc Physics)

- M.Sc. (Physics) (In Central and State Universities and Colleges)
- M.Sc. Physics (JAM in IISc ,IITs and NITs)
- M.Sc. (In State and Central Universities)
- Medical Physics
- Materials Science
- Energy
- Earth Sciences
- Space science
- Oceanography
- Remote sensing
- Electronics
- Photonics
- Optoelectronics
- Acoustics
- Applied electronics
- Astronomy and Astrophysics
- Nanoscience and Nanotechnology
- Biostatistics
- Bio informatics
- Vacuum sciences

Opportunities after B.Sc. Physics

Jobs in Government Sector

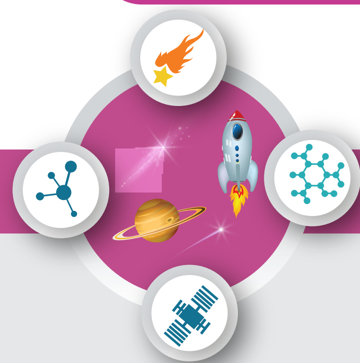
- Indian Forest Services
- Scientist Job in ISRO, DRDO, CSIR labs
- Union Public Service Commission
- Staff selection commission
- Indian Defence services etc.
- Public sector Bank
- State PCS
- Grade III & Compiler Post
- Tax Assistant
- Statistical Investigator
- Combined Higher Secondary
- Combined Graduate Level Exam etc.

Financial assistance to pursue higher education

Scholarships for graduate and post graduate courses

- International Olympiad: for getting stipend for Higher Education in Science and Mathematics
- DST – INSPIRE Scholarships (for UG and PG)
- DST – INSPIRE Fellowships (for Ph.D)
- UGC National Fellowship (for Ph.D)
- Indira Gandhi Fellowship for Single girl child (for UG and PG)
- Moulana Azad Fellowship for minorities (for Ph.D)
- In addition various fellowships for SC/ST/PWD, OBC etc are available.
- Visit website of University Grants Commission (UGC) and Department of Science and Technology (DST)

Institutes in india to pursue research in physics



After completing Post Graduate course in Physics (M.Sc Physics)

Topics of Research

- Quantum Physics and Quantum Optics
- Astrophysics, Astronomy
- String theory, Quantum gravity
- Mathematical Physics, Statistical Mechanics
- Quantum Field Theory
- Particle Physics and Quantum Thermodynamics
- Quantum information theory
- Condensed Matter Physics, Materials Science
- Electro magnetic Theory
- Black Holes, Cosmology
- Crystal Growth, Crystallography
- Spectroscopy, Atomic, Molecular and Optical Physics
- Nano Science and Nanotechnology
- Energy and Environment Studies
- Biophysics, Medical Physics
- Cryptography, Spintronics
- Optics and Photonics
- Meteorology and Atmospheric Science

Research Institutions in various areas of science

Name of the Institution	Website
Indian Institute of Science (IISc) Bangalore	www.iisc.ac.in
Raman Research Institute (RRI) Bangalore	www.rri.res.in
Institute of Mathematical Sciences (IMSc) Chennai	www.imsc.res.in
Indian Association for Cultivation of Science (IACS) Calcutta	www.iacs.res.in
Chennai Mathematical Institute (CMI) Chennai	www.cmi.ac.in
Tata Institute of Fundamental Research (TIFR) Mumbai	www.tifr.res.in
Bhaba Atomic Research centre (BARC) Mumbai	www.barc.gov.in
SN Bose centre Basic Natural science Calcutta	www.bose.res.in
Indian Institute of Space Science and Technology (IIST) Trivandrum	www.iist.ac.in
Physics Research Laboratory (PRL) Ahmedabad	www.prl.res.in
Indian Institute of Astrophysics (IIA) Bangalore	www.iiap.res.in
Institute of Physics (IOP) Bhubaneswar	www.iopb.res.in
Institute for Plasma Research (IPR) Gujarat	www.ipr.res.in
Inter university centre for Astronomy and Astrophysics (IUCAA) Pune	www.iucaa.in
Indira Gandhi centre for Atomic Research (IGCAR), Kalpakkam	www.igcar.gov.in
Hyderabad central university, Hyderabad	www.uohyd.ac.in
Delhi University, Delhi	www.du.ac.in
Mumbai University, Mumbai	www.mu.ac.in
SavithiribaiPhule Pune university, Pune	www.unipune.ac.in
National Institute of Science Education and Research (NISER), Bhubaneshwar	www.niser.ac.in
IISER Educational Institutions	www.iiseradmission.in
Indian Institute of Technology in various places (IIT's)	www.iitm.ac.in
National Institute of Technology (NITs)	www.nitt.edu
Jawaharlal Nehru University (JNU)	www.jnu.ac.in
Central Universities	www.ugc.ac.in
State Universities	https://www.ugc.ac.in
CSIR – Academy (National laboratories, Delhi,	
Hyderabad, Trivandrum, Chennai, Calcutta etc)	

UNIT 1

NATURE OF PHYSICAL WORLD AND MEASUREMENT

"Education is not the learning of facts, but the training of the mind to think" – **Albert Einstein**

Learning Objectives

In this unit, the student is exposed to

- excitement generated by the discoveries in Physics
- an understanding of physical quantities of importance
- different system of units
- an understanding of errors and corrections in physics measurements
- the importance of significant figures
- usage of dimensions to check the homogeneity of physical quantities

1.1 SCIENCE—INTRODUCTION

The word **'science'** has its root in the Latin verb ***scientia***, meaning ***"to know"***. In Tamil language, it is **'அறிவியல்'** (Ariviyal) meaning **'knowing the truth'.** The human mind is always curious to know and understand different phenomena like the bright celestial objects in nature, cyclic changes in the seasons, occurrence of rainbow, etc. The inquisitive mind looks for meaningful patterns and relations in such phenomena. Today's modern science and technology is an offshoot of the understanding of nature. Science is the systematic organization of knowledge gained through observation, experimentation and logical reasoning. The knowledge of science dealing with non-living things is physical science (Physics and Chemistry), and that dealing with living things is biological science (Botany, Zoology etc.).

Curiosity-driven observations of natural happenings was the origin of science. The word **'science'** was coined only in the 19th century. Natural philosophy was the earlier name given to science, when ancient civilization knew and practised astronomy, chemistry, human physiology and agriculture. Oral communication was the mode of conveying knowledge when writing systems were not yet developed. One of the oldest forerunners of scientific advancements, from astronomy to medicine, were the Egyptians. Scientific and mathematical excellence in India dates back to prehistoric human activity in the Indus Valley Civilization (3300 – 1300 BC(BCE).

According to part IV Article 51A (h) of Indian Constitution *"It shall be the duty of every citizen of India to develop scientific temper, humanism and spirit of inquiry and reform"*. This is the aim of our Science Education.

1.1.1 The Scientific Method

The scientific method is a step-by-step approach in studying natural phenomena and establishing laws which govern these phenomena. Any scientific method involves the following general features.

(i) Systematic observation
(ii) Controlled experimentation
(iii) Qualitative and quantitative reasoning
(iv) Mathematical modeling
(v) Prediction and verification or falsification of theories

Example

Consider a metalic rod being heated. When one end of the rod is heated, heat is felt at the other end. The following questions can be asked on this observation

a) What happens within the rod when it is heated?
b) How does the heat reach the other end?
c) Is this effect true for all materials?
d) If heat flows through the material, is it possible to visualize heat?

The process of finding the answers to these queries is scientific investigation.

The basic phenomenon of heat is discussed in unit 8.

DO YOU KNOW?

The name Physics was introduced by Aristotle in the year 350 BC

1.2 PHYSICS - INTRODUCTION

The word 'physics' is derived from the Greek word ***"Fusis",*** meaning nature. The study of nature and natural phenomena is dealt within physics. Hence physics is considered as the most basic of all sciences.

Unification and ***Reductionism*** are the two approaches in studying physics. Attempting to explain diverse physical phenomena with a few concepts and laws is unification. For example, Newton's universal law of gravitation (in unit 6) explains the motion of freely falling bodies towards the Earth, motion of planets around the Sun, motion of the Moon around the Earth, thus unifying the fundamental forces of nature.

An attempt to explain a macroscopic system in terms of its microscopic constituents is ***reductionism***. For example, thermodynamics (unit 8) was developed to explain macroscopic properties like temperature, entropy, etc., of bulk systems. The above properties have been interpreted in terms of the molecular constituents (microscopic) of the bulk system by kinetic theory (unit 9) and statistical mechanics.

1.2.1 Branches of Physics

Physics as a fundamental science helps to uncover the laws of nature. The language of its expression is mathematics. In ancient times, humans lived with nature – their lifestyles were integrated with nature. They could understand the signals from the movement of the stars and other celestial bodies. They could determine the time to sow and reap by watching the sky. Thus, astronomy and mathematics were the first disciplines to be developed. The chronological development of various branches of physics is presented in Appendix A1.1. The various branches of physics are schematically shown in figure 1.1. The essential focus of different areas is given in Table 1.1.

Some of the fundamental concepts of basic areas of physics are discussed in higher secondary first year physics books volume 1 and 2. Mechanics is covered in unit 1 to 6. Unit 1 gives an idea of the development of physics along with discussion on basic elements such as measurement, units etc. Unit 2 gives the basic mathematics needed to express the impact of physical principles and their governing laws. The impact of forces acting on objects in terms of the fundamental laws of motion of Newton are very systematically covered in unit 3. Work and energy which are the basic parameters of investigation of the mechanical world are presented in unit 4. Unit 5 deals with the mechanics of rigid bodies (in contrast, objects are viewed as point objects in units

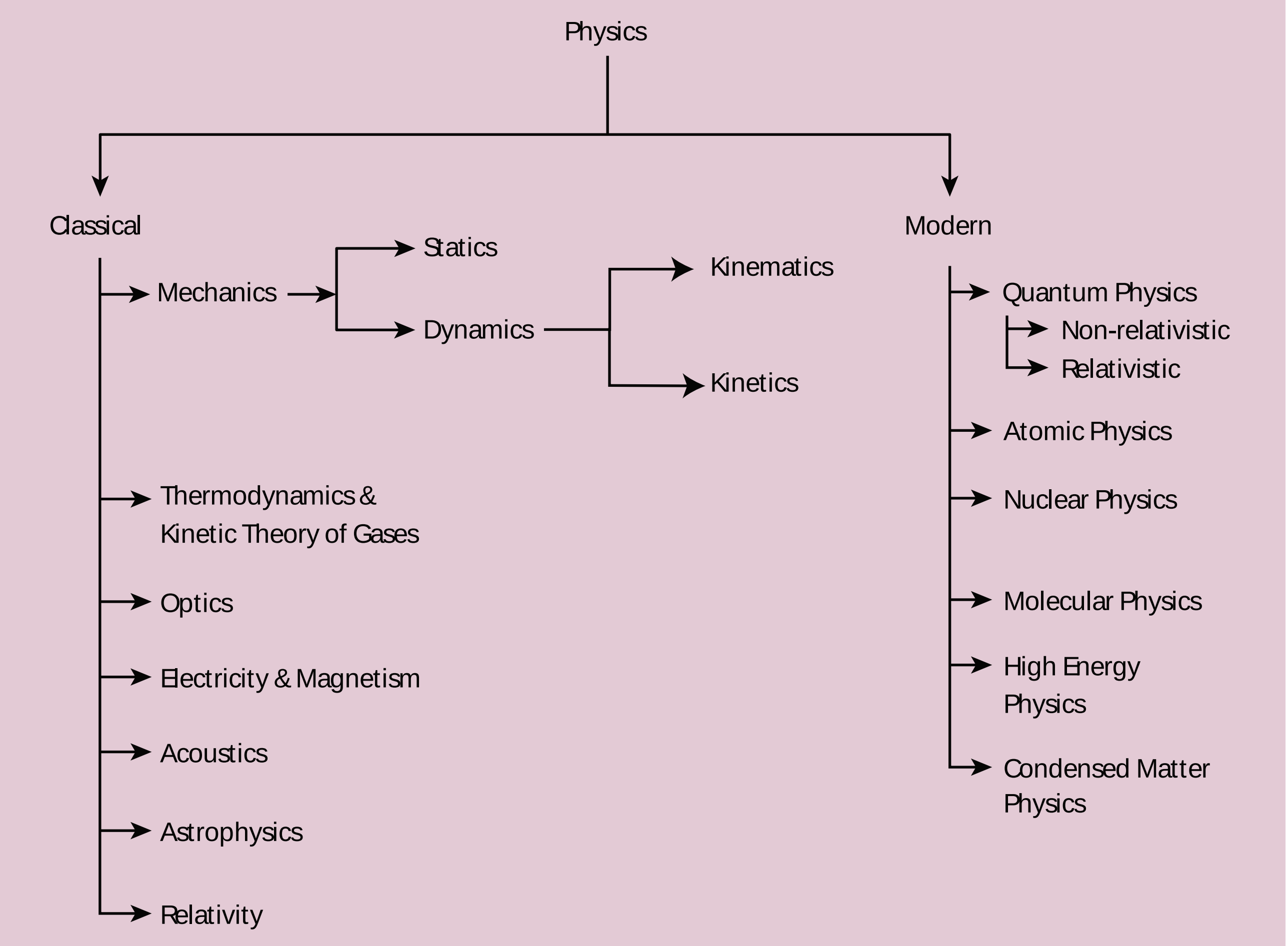


Figure 1.1 Branches of Physics

Table 1.1 Branches of Physics

Classical Physics		**Refers to traditional physics that was recognized and developed before the beginning of the 20th century**
	Branch	**Major focus**
1.	Classical mechanics	The study of forces acting on bodies whether at rest or in motion
2.	Thermodynamics	The study of the relationship between heat and other forms of energy
3.	Optics	The study of light
4.	Electricity and magnetism	The study of electricity and magnetism and their mutual relationship
5.	Acoustics	The study of the production and propagation of sound waves
6.	Astrophysics	The branch of physics which deals with the study of the physics of astronomical bodies
7.	Relativity	One of the branches of theoretical physics which deals with the relationship between space, time and energy particularly with respect to objects moving in different ways .
Modern Physics		**Refers to the concepts in physics that have surfaced since the beginning of the 20th century.**
1.	*Quantum mechanics	The study of the discrete nature of phenomena at the atomic and subatomic levels
2.	Atomic physics	The branch of physics which deals with the structure and properties of the atom
3.	Nuclear physics	The branch of physics which deals with the structure, properties and reaction of the nuclei of atoms.
4.	Condensed matter physics	The study of the properties of condensed materials (solids, liquids and those intermediate between them and dense gas). It branches into various sub-divisions including developing fields such as nano science, photonics etc. It covers the basics of materials science, which aims at developing new material with better properties for promising applications.
5.	High energy physics	The study of the nature of the particles.

*Quantum mechanics is a broader approach; classical results can be reproduced in quantum mechanics also. Detailed explanation is beyond the scope of this book.

3 and 4). The basics of gravitation and its consequences are discussed in unit 6. Older branches of physics such as different properties of matter are discussed in unit 7. The impact of heat and investigations of its consequences are covered in units 8 and 9. Important features of oscillations and wave motion are covered in units 10 and 11.

1.2.2 Scope and Excitement of Physics

Discoveries in physics are of two types; accidental discoveries and well-analysed research outcome in the laboratory based on intuitive thinking and prediction. For example, magnetism was accidentally observed but the reason for this strange behavior of magnets was later analysed theoretically. This analysis revealed the underlying phenomena of magnetism. With this knowledge, artificial magnets were prepared in the laboratories. Theoretical predictions are the most important contribution of physics to the developments in technology and medicine. For example, the famous equation of Albert Einstein, $E=mc^2$ was a theoretical prediction in 1905 and experimentally proved in 1932 by Cockcroft and Walton. Theoretical predictions aided with recent simulation and computation procedures are widely used to identify the most suited materials for robust applications. The pharmaceutical industry uses this technique very effectively to design new drugs. Biocompatible materials for organ replacement are predicted using quantum prescriptions of physics before fabrication. Thus, experiments and theory work hand in hand complimenting one another.

Physics has a huge scope as it covers a tremendous range of magnitude of various physical quantities (length, mass, time, energy etc). It deals with systems of very large magnitude as in astronomical phenomena as well as those with very small magnitude involving electrons and protons.

- Range of time scales: astronomical scales to microscopic scales, 10^{18}s to 10^{-22}s.
- Range of masses: from heavenly bodies to electron, 10^{55} kg (mass of known observable universe) to 10^{-31} kg (mass of an electron) [the actual mass of an electron is 9.11×10^{-31} kg].

The study of physics is not only educative but also exciting in many ways.

- A small number of basic concepts and laws can explain diverse physical phenomena.
- The most interesting part is the designing of useful devices based on the physical laws.

  For example i) use of robotics ii) journey to Moon and to nearby planets with controls from the ground iii) technological advances in health sciences etc.
- Carrying out new challenging experiments to unfold the secrets of nature and in verifying or falsifying the existing theories.
- Probing and understanding the science behind natural phenomena like the eclipse, and why one feels the heat when there is a fire? (or) What causes the wind, etc.

In today's world of technological advancement, the building block of all engineering and technical education is physics which is explained with the help of mathematical tools.

1.3 PHYSICS IN RELATION TO TECHNOLOGY AND SOCIETY

Technology is the application of the principles of physics for practical purposes. The application of knowledge for practical purposes in various fields to invent and produce useful products or to solve problems is known as technology. Thus, physics and technology can both together impact our society directly or indirectly. For example,

i. Basic laws of electricity and magnetism led to the discovery of wireless communication technology which has shrunk the world with effective communication over large distances.
ii. The launching of satellite into space has revolutionized the concept of communication.
iii. Microelectronics, lasers, computers, superconductivity and nuclear energy have comprehensively changed the thinking and living style of human beings.

Physics being a fundamental science has played a vital role in the development of all other sciences. A few examples:

1. **Physics in relation to Chemistry:** In physics, we study the structure of atom, radioactivity, X-ray diffraction etc. Such studies have enabled researchers in chemistry to arrange elements in the periodic table on the basis of their atomic numbers. This has further helped to know the nature of valency, chemical bonding and to understand the complex chemical structures. Inter-disciplinary branches like Physical chemistry and Quantum chemistry play important roles here.
2. **Physics in relation to biology:** Biological studies are impossible without a microscope designed using physics principles. The invention of the electron microscope has made it possible to see even the structure of a cell. X-ray and neutron diffraction techniques have helped us to understand the structure of nucleic acids, which help to control vital life processes. X-rays are used for diagnostic purposes. Radio-isotopes are used in radiotherapy for the cure of cancer and other diseases. In recent years, biological processes are being studied from the physics point of view.
3. **Physics in relation to mathematics:** Physics is a quantitative science. It is most closely related to mathematics as a tool for its development.
4. **Physics in relation to astronomy:** Astronomical telescopes are used to study the motion of planets and other heavenly bodies in the sky. Radio telescopes have enabled the astronomers to observe distant points of the universe. Studies of the universe are done using physical principles.
5. **Physics in relation to geology:** Diffraction techniques help to study the crystal structure of various rocks. Radioactivity is used to estimate the age of rocks, fossils and the age of the Earth.
6. **Physics in relation to oceanography:** Oceanographers seek to understand the physical and chemical processes of the

oceans. They measure parameters such as temperature, salinity, current speed, gas fluxes, chemical components.

7. **Physics in relation to psychology:** All psychological interactions can be derived from a physical process. The movements of neurotransmitters are governed by the physical properties of diffusion and molecular motion. The functioning of our brain is related to our underlying wave-particle dualism.

Nature teaches true science with physics as an efficient tool. Science and technology should be used in a balanced manner so that they do not become weapons to destroy nature which taught us science. Global warming and other negative impacts of technology need to be checked. Safe science with moderate and appropriate use of technology is the need of this century.

The scope and opportunities for higher education in physics and various fellowships offered is given in the beginning of the book.

8TULO

1.4 MEASUREMENT

"When you can measure what you are speaking about and can express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind" - **Lord Kelvin**

The comparison of any physical quantity with its standard unit is known as measurement.

Measurement is the basis of all scientific studies and experimentation. It plays an important role in our daily life. Physics is a quantitative science and physicists always deal with numbers which are the measurement of physical quantities.

1.4.1 Definition of Physical Quantity

Quantities that can be measured, and in terms of which, laws of physics are described are called physical quantities. Examples are length, mass, time, force, energy, etc.

1.4.2 Types of Physical Quantities

Physical quantities are classified into two types. They are fundamental and derived quantities.

Fundamental or base quantities are quantities which cannot be expressed in terms of any other physical quantities. These are length, mass, time, electric current, temperature, luminous intensity and amount of substance.

Quantities that can be expressed in terms of fundamental quantities are called derived quantities. For example, area, volume, velocity, acceleration, force, etc.

1.4.3 Definition of Unit and its Types

The process of measurement is basically a process of comparison. To measure a quantity, we always compare it with some reference standard. For example, when we state that a rope is 10 meter long, it is to say that it is 10 times as long as an object whose length is defined as 1 metre. Such a standard is known as the unit of the quantity. Here 1 metre is the unit of the quantity 'length'.

An arbitrarily chosen standard of measurement of a quantity, which is accepted internationally is called unit of the quantity.

The units in which the fundamental quantities are measured are called *fundamental or base units* and the units of measurement of all other physical quantities, which can be obtained by a suitable multiplication or division of powers of fundamental units, are called *derived units*.

1.4.4 Different types of Measurement Systems

A complete set of units which is used to measure all kinds of fundamental and derived quantities is called a system of units. Here are the common system of units used in mechanics:

(a) **the f.p.s. system** is the British Engineering system of units, which uses ***foot***, ***pound*** and ***second*** as the three basic units for measuring length, mass and time respectively.

(b) **The c.g.s system** is the Gaussian system, which uses ***centimeter, gram*** and ***second*** as the three basic units for measuring length, mass and time respectively.

(c) **The m.k.s system** is based on ***metre, kilogram*** and ***second*** as the three basic units for measuring length, mass and time respectively.

> **DO YOU KNOW?**
> The cgs, mks and SI are metric or decimal system of units. The fps system is not a metric system.

1.4.5 SI unit System

The system of units used by scientists and engineers around the world is commonly called ***the metric system*** but, since 1960, it has been known officially as the International System, or SI (the abbreviation for its French name, Système International). The SI with a standard scheme of symbols, units and abbreviations, were developed and recommended by the General Conference on Weights and Measures in 1971 for international usage in scientific, technical, industrial and commercial work. The advantages of the SI system are,

i) This system makes use of only one unit for one physical quantity, which means a rational system of units.

ii) In this system, all the derived units can be easily obtained from basic and supplementary units, which means it is a coherent system of units.

iii) It is a metric system which means that multiples and submultiples can be expressed as powers of 10.

In SI, there are seven fundamental units as given in Table 1.2

Table 1.2 SI Base Quantities and Units

Base Quantity	SI Units		
	Unit	**Symbol**	**Definition**
Length	metre	m	One metre is the length of the path travelled by light in vacuum in 1/299,792,458 of a second (1983)
Mass	kilogram	kg	One kilogram is the mass of the prototype cylinder of platinum iridium alloy (whose height is equal to its diameter), preserved at the International Bureau of Weights and Measures at Serves, near Paris, France. (1901)
Time	second	s	One second is the duration of 9,192,631,770 periods of radiation corresponding to the transition between the two hyperfine levels of the ground state of Cesium-133 atom.(1967)
Electric current	ampere	A	One ampere is the constant current, which when maintained in each of the two straight parallel conductors of infinite length and negligible cross section, held one metre apart in vacuum shall produce a force per unit length of 2×10^{-7} N/m between them. (1948)
Temperature	kelvin	K	One kelvin is the fraction $\left(\frac{1}{273.16}\right)$ of the thermodynamic temperature of the triple point* of the water. (1967)
Amount of substance	mole	mol	One mole is the amount of substance which contains as many elementary entities as there are atoms in 0.012 kg of pure carbon-12. (1971)
Luminous intensity	candela	cd	One candela is the luminous intensity in a given direction, of a source that emits monochromatic radiation of frequency 5.4×10^{14} Hz and that has a radiant intensity of $\frac{1}{683}$ watt/steradian in that direction. (1979)

* Triple point of water is the temperature at which saturated vapour, pure water and melting ice are all in equilibrium. The triple point temperature of water is 273.16K

Table 1.3 lists some of the derived quantities and their units.

Table 1.3 Derived Quantities and their Units

Physical quantity	Expression	Unit
Plane angle	arc / radius	rad
Solid angle	surface area/radius^2	sr
Area	length $\times$ breadth	m^2
Volume	area $\times$ height	m^3
Velocity	displacement / time	$m\ s^{-1}$
Acceleration	velocity / time	$m\ s^{-2}$
Angular velocity	angular displacement / time	$rad\ s^{-1}$
Angular acceleration	angular velocity / time	$rad\ s^{-2}$
Density	mass / volume	$kg\ m^{-3}$
Linear momentum	mass $\times$ velocity	$kg\ m\ s^{-1}$
Moment of inertia	mass $\times$ (distance)^2	$kg\ m^2$
Force	mass $\times$ acceleration	$kg\ m\ s^{-2}$ or N
Pressure	force / area	$N\ m^{-2}$ or Pa
Energy (work)	force $\times$ distance	N m or J
Power	Work / time	$J\ s^{-1}$ or watt (W)
Impulse	force $\times$ time	N s
Surface tension	force / length	$N\ m^{-1}$
Moment of force (torque)	force $\times$ distance	N m
Electric charge	current $\times$ time	A s or C
Current density	current / area	$A\ m^{-2}$
Magnetic induction	force / (current $\times$ length)	$N\ A^{-1}\ m^{-1}$ or tesla
Force constant	force / displacement	$N\ m^{-1}$
Plank's constant	energy of photon / frequency	J s
Specific heat (S)	heat energy / (mass $\times$ temperature)	$J\ kg^{-1}\ K^{-1}$
Boltzmann constant (k)	energy/temperature	$J\ K^{-1}$

Note:

π radian = 180°

$$1 \text{ radian} = \frac{180°}{\pi} = \frac{180° \times 7}{22} = 57.27°$$

Also, 1° (degree of arc) = 60′ (minute of arc) and 1′ (minute of arc) = 60″ (seconds of arc)

Relations between radian, degree and minutes:

$$1° = \frac{\pi}{180} \text{ rad} = 1.744 \times 10^{-2} \text{ rad}$$

$$\therefore 1' = \frac{1°}{60} = \frac{1.744 \times 10^{-2}}{60} = 2.906 \times 10^{-4}\, rad$$

$$\approx 2.91 \times 10^{-4}\, rad$$

$$\therefore 1'' = \frac{1°}{3600} = \frac{1.744 \times 10^{-2}}{3600} = 4.844 \times 10^{-6}\, rad$$

$$\approx 4.84 \times 10^{-6}\, rad$$

1.5 MEASUREMENT OF BASIC QUANTITIES

1.5.1 Measurement of length

The concept of length in physics is related to the concept of distance in everyday life. Length is defined as the distance between any two points in space. The SI unit of length is metre. The objects of our interest vary widely in sizes. For example, large objects like the galaxy, stars, Sun, Earth, Moon etc., and their distances constitute a ***macrocosm***. It refers to a large world, in which both objects and distances are large. On the contrary, objects like molecules, atoms, proton, neutron, electron, bacteria etc., and their distances constitute ***microcosm***, which means a small world in which both objects and distances are small-sized.

The Radian (rad): *One radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.*
The Steradian (sr): *One steradian is the solid angle subtended at the centre of a sphere, by that surface of the sphere, which is equal in area, to the square of radius of the sphere*

Distances ranging from 10^{-5} m to 10^{2} m can be measured by direct methods. For example, a metre scale can be used to measure the distance from 10^{-3} m to 1 m, vernier calipers up to 10^{-4} m, a screw gauge up to 10^{-5} m and so on. The atomic and astronomical distances cannot be measured by any of the above mentioned direct methods. Hence, to measure the very small and the very large distances, indirect methods have to be devised and used. In Table 1.4, a list of powers of 10 (both positive and negative powers) is given. Prefixes for each power are also mentioned. These prefixes are used along with units of length, and of mass.

DO YOU KNOW?

The supplementary quantities of plane and solid angle were converted into Derived quantities in 1995 (GCWM)

Table 1.4 Prefixes for Powers of Ten

Multiple	Prefix	Symbol	Sub multiple	Prefix	Symbol
10^{1}	deca	da	10^{-1}	deci	d
10^{2}	hecto	h	10^{-2}	centi	c
10^{3}	kilo	k	10^{-3}	milli	m
10^{6}	mega	M	10^{-6}	micro	μ
10^{9}	giga	G	10^{-9}	nano	n
10^{12}	tera	T	10^{-12}	pico	p
10^{15}	peta	P	10^{-15}	femto	f
10^{18}	exa	E	10^{-18}	atto	a
10^{21}	zetta	Z	10^{-21}	zepto	z
10^{24}	yotta	Y	10^{-24}	yocto	y

i) Measurement of small distances: screw gauge and vernier caliper

Screw gauge: The screw gauge is an instrument used for measuring accurately the dimensions of objects up to a maximum of about 50 mm. The principle of the instrument is the magnification of linear motion using the circular motion of a screw. The least count of the screw gauge is 0.01 mm

Vernier caliper: A vernier caliper is a versatile instrument for measuring the dimensions of an object namely diameter of a hole, or a depth of a hole. The least count of the vernier caliper is 0.01 cm

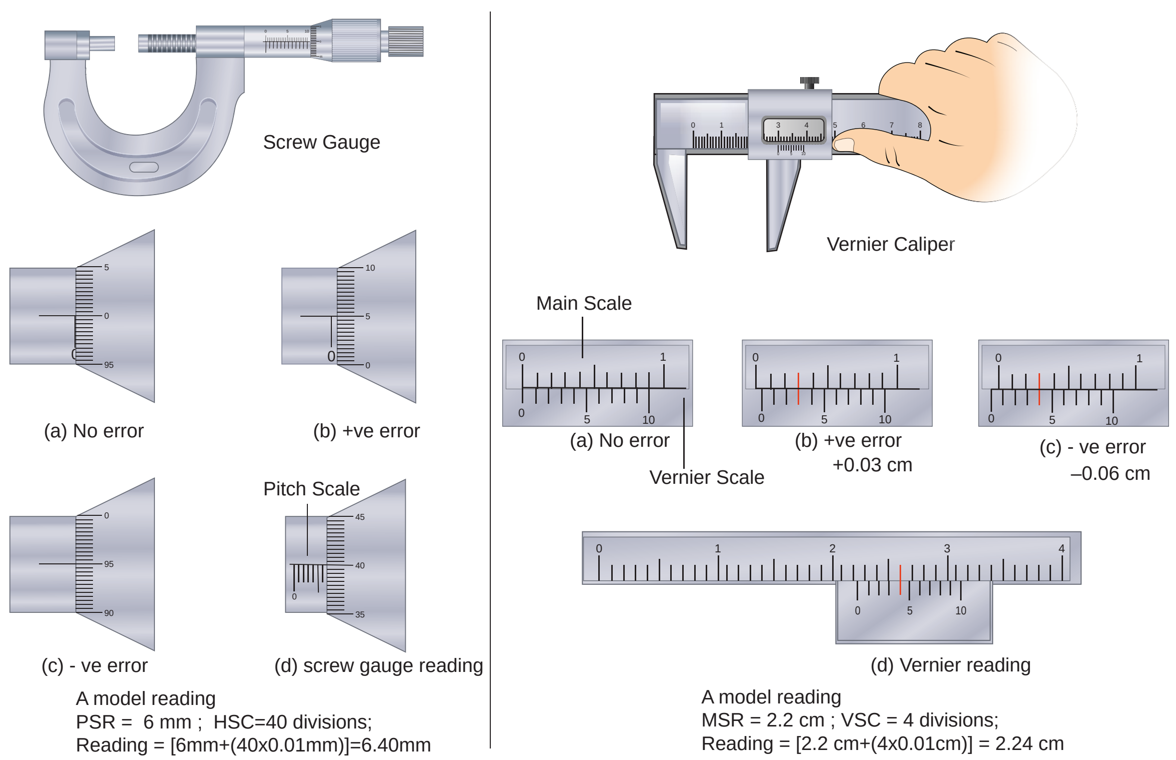


Figure 1.2 Screw gauge and vernier caliper with errors

ii) Measurement of large distances

For measuring larger distances such as the height of a tree, distance of the Moon or a planet from the Earth, some special methods are adopted. Triangulation method, parallax method and radar method are used to determine very large distances.

Triangulation method for the height of an accessible object

Let AB = h be the height of the tree or tower to be measured. Let C be the point of observation at distance x from B. Place a range finder at C and measure the angle of elevation, $\angle ACB = \theta$ as shown in Figure 1.3.

From right angled triangle ABC,

$$\tan\theta = \frac{AB}{BC} = \frac{h}{x}$$

(or)

$$\text{height } h = x \tan\theta$$

Knowing the distance x, the height h can be determined.

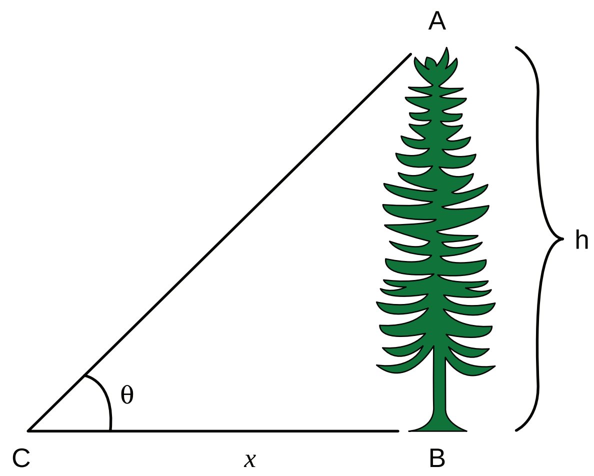


Figure 1.3 Triangulation method

Range and order of lengths of various objects are listed in Table 1.5

EXAMPLE 1.1

From a point on the ground, the top of a tree is seen to have an angle of elevation 60°. The distance between the tree and a point is 50 m. Calculate the height of the tree?

Solution

$$\text{Angle } \theta = 60°$$

The distance between the tree and a point $x = 50$ m

Height of the tree (h) = ?

For triangulation method $\tan\theta = \frac{h}{x}$

$$h = x \tan\theta$$
$$= 50 \times \tan 60°$$
$$= 50 \times 1.732$$
$$h = 86.6 \text{ m}$$

The height of the tree is 86.6 m.

Parallax method

Very large distances, such as the distance of a planet or a star from the Earth can be measured by the parallax method. *Parallax is the name given to the apparent change in the position of an object with respect to the background, when the object is seen from two different positions.* The distance between the two positions (i.e., points of observation) is called the basis (b). Consider any object at the location O (Figure 1.4)

Let L and R represent the positions of the left and right eyes of the observer respectively.

The object (O) is viewed with the left eye (L) keeping the right eye closed and the same object (O) is viewed with the right eye (R) keeping the left eye closed.

UNIT-1(XI-Physics_Vol-1).indd 13 05-01-2022 22:11:40

In Figure 1.4, LO and RO are the lines drawn from the positions of the left and right eyes to the object. These two lines make an angle θ at O. This angle θ is called the angle of parallax.

OL and OR are considered as the radii (x) of a circle. For astronomical calculation, the distance LR = b (basis) can be treated as an arc of this circle, then

OL = OR = x

as LR = b

$$\theta = \frac{b}{x}$$

Knowing b and θ, x can be calculated which is approximately the distance of the object from the observer.

If the object is the Moon or any near by star, then the angle θ will be too small due to the large astronomical distance and the place of observation. In this case, the two points of observation should be sufficiently spaced on the surface of the Earth.

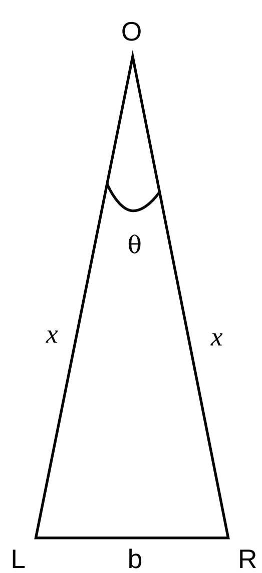


Figure 1.4 Parallax method

Determination of distance of Moon from Earth

In Figure 1.5, C is the centre of the Earth. A and B are two diametrically opposite places on the surface of the Earth. From A and B, the parallaxes θ_1 and θ_2 respectively of Moon M with respect to some distant star are determined with the help of an astronomical telescope. Thus, the total parallax of the Moon subtended on Earth $\angle AMB = \theta_1 + \theta_2 = \theta$.

If θ is measured in radians, then $\theta = \frac{AB}{AM}$; AM≈MC (AM is approximately equal to MC)

$\theta = \frac{AB}{MC}$ or $MC = \frac{AB}{\theta}$; Knowing the values of AB and θ, we can calculate the distance MC of Moon from the Earth.

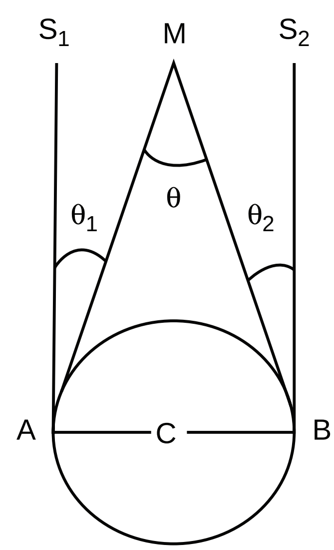


Figure 1.5 Parallax method: determination of distance of Moon from Earth

EXAMPLE 1.2

The Moon subtends an angle of 1° 55′ at the base line equal to the diameter of the Earth. What is the distance of the Moon from the Earth? (Radius of the Earth is $6.4 \times 10^6\,m$)

Solution

$$\text{angle } \theta = 1^\circ\ 55' = 115'$$
$$= (115 \times 60)'' \times (4.85 \times 10^{-6}) \text{ rad}$$
$$= 3.34 \times 10^{-2} \text{ rad}$$
$$\text{since } 1'' = 4.85 \times 10^{-6} \text{ rad}$$

Radius of the Earth $= 6.4 \times 10^6 m$

From the Figure 1.5, AB is the diameter of the Earth (b)= $2 \times 6.4 \times 10^6 m$ Distance of the Moon from the Earth $x = ?$

$$x = \frac{b}{\theta} = \frac{2 \times 6.4 \times 10^6}{3.34 \times 10^{-2}}$$
$$x = 3.83 \times 10^8 \text{m}$$

RADAR method

The word RADAR stands for radio detection and ranging. A radar can be used to measure accurately the distance of a nearby planet such as Mars. In this method, radio waves are sent from transmitters which, after reflection from the planet, are detected by the receiver. By measuring, the time interval (t) between the instants the radio waves are sent and received, the distance of the planet can be determined as

Speed = distance travelled / time taken
(Speed is explained in unit 2)
Distance(d) = Speed of radio waves × time taken

$$d = \frac{v \times t}{2}$$

where v is the speed of the radio wave. As the time taken (t) is for the distance covered during the forward and backward path of the radio waves, it is divided by 2 to get the actual distance of the object. This method can also be used to determine the height, at which an aeroplane flies from the ground.

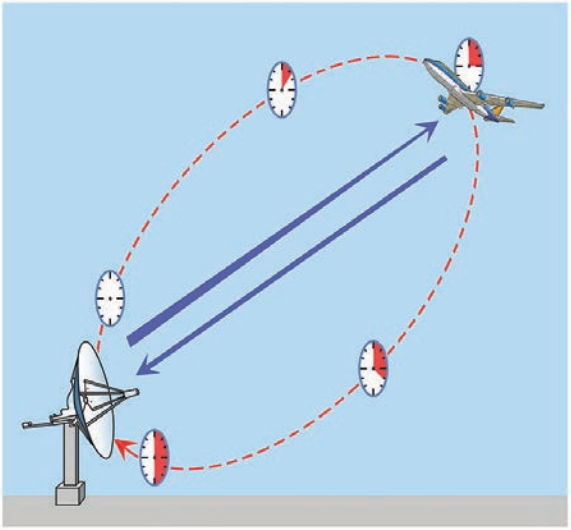


Figure 1.6 RADAR method

EXAMPLE 1.3

A RADAR signal is beamed towards a planet and its echo is received 7 minutes later. If the distance between the planet and the Earth is 6.3×10^{10} m. Calculate the speed of the signal?

Solution

The distance of the planet from the Earth $d = 6.3 \times 10^{10}$ m

Time $t = 7$ minutes $= 7 \times 60$ s.
the speed of signal $v = ?$

The speed of signal

$$v = \frac{2d}{t} = \frac{2 \times 6.3 \times 10^{10}}{7 \times 60} = 3 \times 10^8 ms^{-1}$$

Table 1.5 Range and Order of Lengths

Size of objects and distances	Length (m)
Distance to the boundary of observable universe	10^{26}
Distance to the Andromeda galaxy	10^{22}
Size of our galaxy	10^{21}
Distance from Earth to the nearest star (other than the Sun)	10^{16}
Average radius of Pluto's orbit	10^{12}
Distance of the Sun from the Earth	10^{11}
Distance of Moon from the Earth	10^{8}
Radius of the Earth	10^{7}
Height of the Mount Everest above sea level	10^{4}
Length of a football field	10^{2}
Thickness of a paper	10^{-4}
Diameter of a red blood cell	10^{-5}
Wavelength of light	10^{-7}
Length of typical virus	10^{-8}
Diameter of the hydrogen atom	10^{-10}
Size of atomic nucleus	10^{-14}
Diameter of a proton	10^{-15}

Some Common Practical Units

(i) Fermi = 1 fm = 10^{-15} m

(ii) 1 angstrom = 1 Å = 10^{-10} m

(iii) 1 nanometer = 1 nm = 10^{-9} m

(iv) 1 micron = 1μm = 10^{-6} m

(v) 1 Light year (Distance travelled by light in vacuum in one year) 1 Light Year = 9.467×10^{15} m

(vi) 1 astronomical unit (the mean distance of the Earth from the Sun) 1 AU = 1.496×10^{11} m

(vii) 1 parsec (Parallactic second) (Distance at which an arc of length 1 AU subtends an angle of 1 second of arc)

1 parsec = 3.08×10^{16} m =3.26 light year

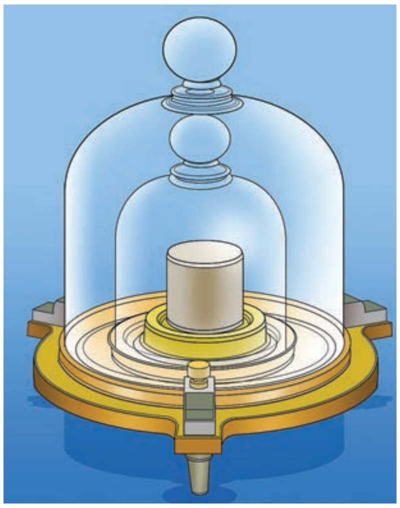


Figure 1.7 The international 1 kg standard of mass, a platinum-iridium (9:1) cylinder 3.9 cm in height and diameter.

Why is the cylinder used in defining kilogram made up of platinum-iridium alloy?

This is because the platinum-iridium alloy is least affected by environment and time.

Chandrasekhar Limit (CSL) is the largest practical unit of mass.

1 CSL = 1.4 times the mass of the Sun

The smallest practical unit of time is Shake.

1 Shake = 10^{-8} s

1.5.2 Measurement of mass

Mass is a property of matter. It does not depend on temperature, pressure and location of the body in space. *Mass of a body is defined as the quantity of matter contained in a body*. The SI unit of mass is kilogram (kg). The masses of objects which we shall study in this course vary over a wide range. These may vary from a tiny mass of electron (9.11×10^{-31}kg) to the huge mass of the known universe ($=10^{55}$ kg). The order of masses of various objects is shown in Table 1.6.

Ordinarily, the mass of an object is determined in kilograms using a common balance like the one used in a grocery shop. For measuring larger masses like that of planets, stars etc., we make use of gravitational methods. For measurement of small masses of atomic/subatomic particles etc., we make use of a mass spectrograph.

Some of the weighing balances commonly used are common balance, spring balance, electronic balance, etc.

Table 1.6 Range of masses

Object	Order of mass (kg)
Electron	10^{-30}
Proton or Neutron	10^{-27}
Uranium atom	10^{-25}
Red blood corpuscle	10^{-14}
A cell	10^{-10}
Dust particle	10^{-9}
Raindrop	10^{-6}
Mosquito	10^{-5}
Grape	10^{-3}
Frog	10^{-1}
Human	10^{2}
Car	10^{3}
Ship	10^{5}
Moon	10^{23}
Earth	10^{25}
Sun	10^{30}
Milky way	10^{41}
Observable Universe	10^{55}

1.5.3 Measurement of Time intervals

"Time flows uniformly forward"
– Sir Issac Newton

"Time is what a clock reads"
– Albert Einstein

A clock is used to measure the time interval. An atomic standard of time, is based on the periodic vibration produced in a Cesium atom. Some of the clocks developed later are electric oscillators, electronic oscillators, solar clock, quartz crystal clock, atomic clock, decay of elementary particles, radioactive dating etc. The order of time intervals are tabulated in Table 1.7.

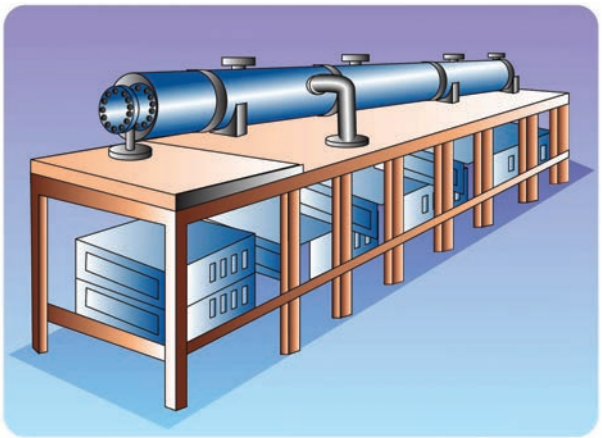


Figure 1.8 The atomic clock, which keeps time on the basis of radiation from cesium atoms is accurate to about three millionths of a second per year.

Table 1.7 Order of Time Intervals

Event	Order of time interval (s)
Lifespan of the most unstable particle	10^{-24}
Time taken by light to cross a distance of nuclear size	10^{-22}
Period of X-rays	10^{-19}
Time period of electron in hydrogen atom	10^{-15}
Period of visible light waves	10^{-15}
Time taken by visible light to cross through a window pane	10^{-8}
Lifetime of an excited state of an atom	10^{-8}
Period of radio waves	10^{-6}
Time period of audible sound waves	10^{-3}
Wink of an eye	10^{-1}
Time interval between two successive heart beats	10^{0}
Travel time of light from Moon to Earth	10^{0}
Travel time of light from Sun to Earth	10^{2}
Halflife time of a free neutron	10^{3}
Time period of a satellite	10^{4}
Time period of rotation of Earth around its axis (one day)	10^{5}
Time period of revolution of Earth around the Sun (one year)	10^{7}
Average life of a human being	10^{9}
Age of Egyptian pyramids	10^{11}
Age of Universe	10^{17}

1.6 THEORY OF ERRORS

The foundation of all experimental science and technology is measurement. The result obtained from any measurement will contain some uncertainty. Such an uncertainty is termed ***error***. Any calculation made using the measured values will also have an error. It is not possible to make exact measurements in an experiment. In measurements, two different terms, accuracy and precision are used and need to be distinguished at this stage. Accuracy refers to how far we are from the true value, and precision refers to how well we measure.

DO YOU KNOW?

In India, the National Physical Laboratory (New Delhi) has the responsibility of maintenance and improvement of physical standards of length, mass, time, etc.

1.6.1 Accuracy and Precision

Let us say, you know your true height is exactly 5′9″. You first measure your height with a yardstick and get the value 5′0″. Your measurement is hence not accurate. Now you measure your height with a laser yardstick and get 5′9″ as the value. Now your measurement is accurate. The true value is also called theoretical value. The level of accuracy required for each application varies greatly. Highly accurate data can be very difficult to produce and compile. For example, if you consistently measure your height as 5′0″ with a yard stick, your measurements are precise. The level of precision required for different applications vary to a great extent. Engineering projects such as road and utility construction require very precise information measured to the millimeter or one-tenth of an inch.

If a measurement is precise, that does not necessarily mean that it is accurate. However, if the measurement is consistently accurate, it is also precise.

For example, if the temperature outside a building is 40°C as measured by a weather thermometer and if the real outside temperature is 40°C, the thermometer is accurate. If the thermometer consistently registers this exact temperature in a row, the thermometer is precise.

Consider another example. Let the temperature of a refrigerator repeatedly measured by a thermometer be given as 10.4°C, 10.2°C, 10.3°C, 10.1°C, 10.2°C, 10.1°C, 10.1°C, 10.1°C. However, if the real temperature inside the refrigerator is 9°C, we say that the thermometer is not accurate (it is almost one degree off the true value), but since all the measured values are close to 10°C, hence it is precise.

A visual example:

Target shooting is an example which explains the difference between accuracy and precision. In Figure 1.9 (a), the shots are focused so as to reach the bull's eye (midpoint), but the arrows have reached only around this point. Hence the shots are not accurate and also not precise.

In Figure 1.9 (b), all the shots are close to each other but not at the central point. Hence the shots are said to be precise but not accurate. In Figure 1.9 (c), the shots are closer and also at the central point. Hence the shots are both precise and accurate.

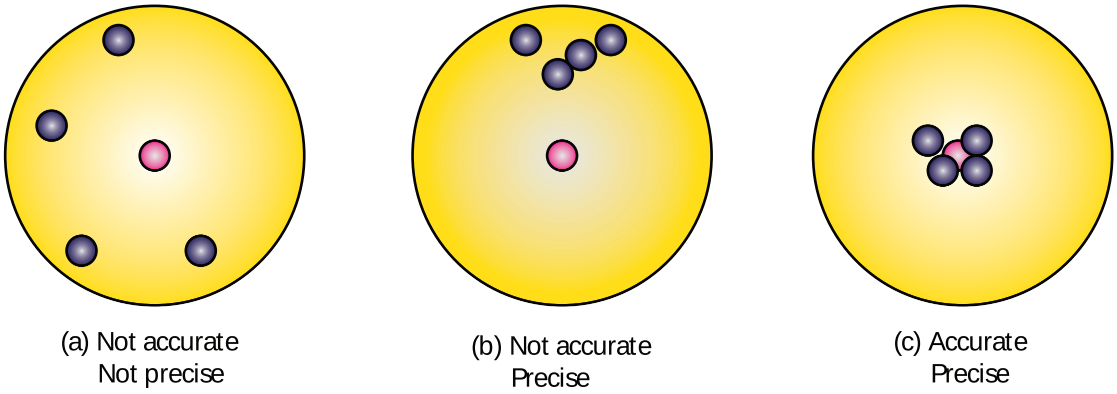


Figure 1.9 Visual example of accuracy and precision

A numerical example

The true value of a certain length is nearly 5.678 cm. In one experiment, using a measuring instrument of resolution 0.1 cm, the measured value is found to be 5.5 cm. In another experiment using a measuring instrument of greater resolution, say 0.01 cm, the length is found to be 5.38 cm. We find that the first measurement is more accurate as it is closer to the true value, but it has lesser precision. On the contrary, the second measurement is less accurate, but it is more precise.

1.6.2 Errors in Measurement

The uncertainty in a measurement is called an error. Random error, systematic error and gross error are the three possible errors.

i) Systematic errors

Systematic errors are reproducible inaccuracies that are consistently in the same direction. These occur often due to a problem that persists throughout the experiment. Systematic errors can be classified as follows

1) Instrumental errors

When an instrument is not calibrated properly at the time of manufacture, instrumental errors may arise. If a measurement is made with a meter scale whose end is worn out, the result obtained will have errors. These errors can be corrected by choosing the instrument carefully.

2) Imperfections in experimental technique or procedure

These errors arise due to the limitations in the experimental arrangement. As an example, while performing experiments with a calorimeter, if there is no proper insulation, there will be radiation losses. This results in errors and to overcome these, necessary correction has to be applied.

3) Personal errors

These errors are due to individuals performing the experiment, may be due to incorrect initial setting up of the experiment or carelessness of the individual making the observation due to improper precautions.

4) Errors due to external causes

The change in the external conditions during an experiment can cause error in measurement. For example, changes in temperature, humidity, or pressure during measurements may affect the result of the measurement.

5) Least count error

Least count is the smallest value that can be measured by the measuring instrument, and the error due to this measurement is least count error. The instrument's resolution hence is the cause of this error. Least count error can be reduced by using a high precision instrument for the measurement.

ii) Random errors

Random errors may arise due to random and unpredictable variations in experimental conditions like pressure, temperature, voltage supply etc. Errors may also be due to personal errors by the observer who performs the experiment. Random errors are sometimes called **"chance error"**. When different readings are obtained by a person every time he repeats the experiment, personal error occurs. For example, consider the case of the thickness of a wire measured using a screw gauge. The readings taken may be different for different trials. In this case, a large number of measurements are made and then the arithmetic mean is taken.

If n number of trial readings are taken in an experiment, and the readings are a_1, a_2, a_3,.................... a_n. The arithmetic mean is

$$a_m = \frac{a_1 + a_2 + a_3 + ...a_n}{n} \quad (1.1)$$

or

$$a_m = \frac{1}{n}\sum_{i=1}^{n} a_i \quad (1.2)$$

Usually this arithmetic mean is taken as the best possible true value of the quantity.

Certain procedures to be followed to minimize experimental errors, along with examples are shown in Table 1.8.

iii) Gross Error

The error caused due to the shear carelessness of an observer is called gross error.

For example

(i) Reading an instrument without setting it properly.

(ii) Taking observations in a wrong manner without bothering about the sources of errors and the precautions.

(iii) Recording wrong observations.

(iv) Using wrong values of the observations in calculations.

These errors can be minimized only when an observer is careful and mentally alert.

Table 1.8 Minimizing Experimental Error

Type of error	Example	How to minimize it
Random error	Suppose you measure the mass of a ring three times using the same balance and get slightly different values. 15.46 g, 15.42 g, 15.44 g	Take more data. Random errors can be evaluated through statistical analysis and can be reduced by averaging over a large number of observations.
Systematic error	Suppose the cloth tape measure that you use to measure the length of an object has been stretched out from years of use. (As a result all of the length measurements are not correct).	Systematic errors are difficult to detect and cannot be analysed statistically, because all of the data is in the same direction. (Either too high or too low)

1.6.3 Error Analysis

i) Absolute Error

The magnitude of difference between the true value and the measured value of a quantity is called absolute error. If a_1, a_2, a_3, ...a_n are the measured values of any quantity 'a' in an experiment performed n times, then the arithmetic mean of these values is called the true value (a_m) of the quantity.

$$a_m = \frac{a_1 + a_2 + a_3 + ... + a_n}{n} \text{ or}$$

$$a_m = \frac{1}{n}\sum_{i=1}^{n} a_i$$

The absolute error in measured values is given by

$$|\Delta a_1| = |a_m - a_1|$$
$$|\Delta a_2| = |a_m - a_2|$$
$$..................$$
$$..................$$
$$|\Delta a_n| = |a_m - a_n|$$

ii) Mean Absolute error

The arithmetic mean of absolute errors in all the measurements is called the mean absolute error.

$$\Delta a_m = \frac{|\Delta a_1| + |\Delta a_2| + |\Delta a_3| + ... + |\Delta a_n|}{n}$$

$$\text{or} = \frac{1}{n}\sum_{i=1}^{n}|\Delta a_i|$$

If a_m is the true value and Δa_m is the mean absolute error then the magnitude of the quantity may lie between $a_m + \Delta a_m$ and $a_m - \Delta a_m$

iii) Relative error

The ratio of the mean absolute error to the mean value is called relative error. This is also called as fractional error. Thus

$$\text{Relative error} = \frac{\textit{Mean absolute error}}{\textit{Mean value}}$$
$$= \frac{\Delta a_m}{a_m}$$

Relative error expresses how large the absolute error is compared to the total size of the object measured. For example, a driver's speedometer shows that his car is travelling at 60 km h^{-1} when it is actually moving at 62 km h^{-1}. Then absolute error of speedometer is 62-60 km h^{-1} = 2 km h^{-1} Relative error of the measurement is 2 km h^{-1} / 62 km h^{-1} = 0.032.

iv) Percentage error

The relative error expressed as a percentage is called percentage error.

$$\text{Percentage error} = \frac{\Delta a_m}{a_m} \times 100\%$$

A percentage error very close to zero means one is close to the targeted value, which is good and acceptable. It is always necessary to understand whether error is due to impression of equipment used or a mistake in the experimentation.

EXAMPLE 1.4

In a series of successive measurements in an experiment, the readings of the period of oscillation of a simple pendulum were found to be 2.63s, 2.56 s, 2.42s, 2.71s and

2.80s. Calculate (i) the mean value of the period of oscillation (ii) the absolute error in each measurement (iii) the mean absolute error (iv) the relative error (v) the percentage error. Express the result in proper form.

Solution

$$t_1 = 2.63\,s,\ t_2 = 2.56\,s,\ t_3 = 2.42\,s,$$
$$t_4 = 2.71\,s,\ t_5 = 2.80\,s$$

(i) $$T_m = \frac{t_1 + t_2 + t_3 + t_4 + t_5}{5}$$
$$= \frac{2.63 + 2.56 + 2.42 + 2.71 + 2.80}{5}$$
$$T_m = \frac{13.12}{5} = 2.624s$$
$T_m = 2.62s$ (Rounded off to 2nd decimal place)

(ii) Absolute error $|\Delta T| = |T_m - t|$

$$|\Delta T_1| = |2.62 - 2.63| = +0.01s$$
$$|\Delta T_2| = |2.62 - 2.56| = +0.06s$$
$$|\Delta T_3| = |2.62 - 2.42| = +0.20s$$
$$|\Delta T_4| = |2.62 - 2.71| = +0.09s$$
$$|\Delta T_5| = |2.62 - 2.80| = +0.18s$$

(iii) Mean absolute error $= \dfrac{\Sigma|\Delta T_i|}{n}$

$$\Delta T_m = \frac{0.01 + 0.06 + 0.20 + 0.09 + 0.18}{5}$$

$\Delta T_m = \dfrac{0.54}{5} = 0.108s = 0.11s$ (Rounded off to 2nd decimal place)

(iv) Relative error:

$$S_T = \frac{\Delta T_m}{T_m} = \frac{0.11}{2.62} = 0.0\,419$$
$$S_T = 0.04$$

(v) Percentage error in T = 0.04 × 100% = 4%

(vi) Time period of simple pendulum = T = (2.62 ± 0.11)s

1.6.4 Propagation of errors

A number of measured quantities may be involved in the final calculation of an experiment. Different types of instruments might have been used for taking readings. Then we may have to look at the errors in measuring various quantities, collectively.

The error in the final result depends on

(i) The errors in the individual measurements

(ii) On the nature of mathematical operations performed to get the final result. So we should know the rules to combine the errors.

The various possibilities of the propagation or combination of errors in different mathematical operations are discussed below:

(i) Error in the sum of two quantities

Let ΔA and ΔB be the absolute errors in the two quantities A and B respectively. Then,

Measured value of A = A ± ΔA

Measured value of B = B ± ΔB

Consider the sum, Z = A + B

The error ΔZ in Z is then given by

$$Z \pm \Delta Z = (A \pm \Delta A) + (B \pm \Delta B)$$
$$= (A + B) \pm (\Delta A + \Delta B)$$
$$= Z \pm (\Delta A + \Delta B)$$
$$\text{(or)} \quad \Delta Z = \Delta A + \Delta B \qquad (1.3)$$

The maximum possible error in the sum of two quantities is equal to the sum of the absolute errors in the individual quantities.

EXAMPLE 1.5

Two resistances R_1 = (100 ± 3) Ω, R_2 = (150 ± 2) Ω, are connected in series. What is their equivalent resistance?

Solution

$$R_1 = 100 \pm 3\Omega;\ R_2 = 150 \pm 2\Omega$$

Equivalent resistance R = ?

Equivalent resistance $R = R_1 + R_2$

$$= (100 \pm 3) + (150 \pm 2)$$
$$= (100 + 150) \pm (3 + 2)$$
$$R = (250 \pm 5)\ \Omega$$

(ii) Error in the difference of two quantities

Let ΔA and ΔB be the absolute errors in the two quantities, A and B, respectively. Then,

Measured value of A = A ± ΔA

Measured value of B = B ± ΔB

Consider the difference, Z = A – B

The error ΔZ in Z is then given by

$$Z \pm \Delta Z = (A \pm \Delta A) - (B \pm \Delta B)$$
$$= (A - B) \pm \Delta A \mp \Delta B$$
$$= Z \pm \Delta A \mp \Delta B$$
$$\text{(or)} \quad \Delta Z = \Delta A + \Delta B \ldots\ldots\ldots\ldots\ldots\ldots (1.4)$$

The maximum error in difference of two quantities is equal to the sum of the absolute errors in the individual quantities.

EXAMPLE 1.6

The temperatures of two bodies measured by a thermometer are t_1 = (20 + 0.5)°C, t_2 = (50 ± 0.5)°C. Calculate the temperature difference and the error therein.

Solution

$t_1 = (20 \pm 0.5)$°C $t_2 = (50 \pm 0.5)$°C

temperature difference t=?

$t = t_2 - t_1 = (50 \pm 0.5) - (20 \pm 0.5)$°C
(Using equation1.4)

$= (50 - 20) \pm (0.5 + 0.5)$

$t = (30 \pm 1)$°C

(iii) Error in the product of two quantities

Let ΔA and ΔB be the absolute errors in the two quantities A, and B, respectively. Consider the product Z = AB

The error ΔZ in Z is given by $Z \pm \Delta Z = (A \pm \Delta A)(B \pm \Delta B)$

$$= (AB) \pm (A\,\Delta B) \pm (B\,\Delta A) \pm (\Delta A \,.\, \Delta B)$$

Dividing L.H.S by Z and R.H.S by AB, we get,

$$1 \pm \frac{\Delta Z}{Z} = 1 \pm \frac{\Delta B}{B} \pm \frac{\Delta A}{A} \pm \frac{\Delta A}{A} . \frac{\Delta B}{B}$$

As ΔA /A, ΔB / B are both small quantities, their product term $\frac{\Delta A}{A}.\frac{\Delta B}{B}$ can be neglected. The maximum fractional error in Z is

$$\frac{\Delta Z}{Z} = \pm\left(\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) \qquad (1.5)$$

The maximum fractional error in the product of two quantities is equal to the sum of the fractional errors in the individual quantities.

[Alternative method is given in Appendix A1.2]

EXAMPLE 1.7

The length and breadth of a rectangle are (5.7 ±0.1) cm and (3.4 ±0.2) cm respectively. Calculate the area of the rectangle with error limits.

Solutions

Length $\ell = (5.7 \pm 0.1)$ cm

Breadth b =(3.4 ± 0.2) cm

Area A with error limit = A ± ΔA = ?

Area A = $\ell \times b = 5.7 \times 3.4 = 19.38 = 19.4\ cm^2$

$$\frac{\Delta A}{A} = \frac{\Delta l}{l} + \frac{\Delta b}{b}$$

$$\Delta A = \left(\frac{\Delta l}{l} + \frac{\Delta b}{b}\right)A$$

$$\Delta A = \left(\frac{0.1}{5.7} + \frac{0.2}{3.4}\right)19.4$$

$$= (0.0175+0.0588) \times 19.4$$

$$= 1.48 = 1.5$$

Area with error limit

A = (19.4±1.5) cm^2

(iv) Error in the division or quotient of two quantities

Let ΔA and ΔB be the absolute errors in the two quantities A and B respectively.

Consider the quotient, $Z = \frac{A}{B}$

The error ΔZ in Z is given by

$$Z \pm \Delta Z = \frac{A \pm \Delta A}{B \pm \Delta B} = \frac{A\left(1 \pm \frac{\Delta A}{A}\right)}{B\left(1 \pm \frac{\Delta B}{B}\right)}$$

$$= \frac{A}{B}\left(1 \pm \frac{\Delta A}{A}\right)\left(1 \pm \frac{\Delta B}{B}\right)^{-1}$$

or $Z \pm \Delta Z = Z\left(1 \pm \frac{\Delta A}{A}\right)\left(1 \mp \frac{\Delta B}{B}\right)$ [using $(1+x)^n \approx 1+nx$, when x<<1]

Dividing both sides by Z, we get,

$$1 \pm \frac{\Delta Z}{Z} = \left(1 \pm \frac{\Delta A}{A}\right)\left(1 \mp \frac{\Delta B}{B}\right)$$

$$= 1 \pm \frac{\Delta A}{A} \mp \frac{\Delta B}{B} \pm \frac{\Delta A}{A}.\frac{\Delta B}{B}$$

As the terms ΔA/A and ΔB/B are small, their product term can be neglected.

The maximum fractional error in Z is given by $\frac{\Delta Z}{Z} = \left(\frac{\Delta A}{A} + \frac{\Delta B}{B}\right)$ (1.6)

The maximum fractional error in the quotient of two quantities is equal to the sum of their individual fractional errors.

(Alternative method is given in Appendix A1.2)

EXAMPLE 1.8

The voltage across a wire is (100 ± 5)V and the current passing through it is (10±0.2) A. Find the resistance of the wire.

Solution

Voltage V = (100±5)V

Current I = (10±0.2) A

Resistance R = ?

Then resistance R is given by Ohm's law, $R = \frac{V}{I}$

$$= \frac{100}{10} = 10\Omega$$

$$\frac{\Delta R}{R} = \left(\frac{\Delta V}{V} + \frac{\Delta I}{I}\right)$$

$$\Delta R = \left(\frac{\Delta V}{V} + \frac{\Delta I}{I}\right)R$$

$$= \left(\frac{5}{100} + \frac{0.2}{10}\right)10$$

$$= (0.05 + 0.02)10$$

$$= 0.07 \times 10 = 0.7$$

The resistance R = (10 ± 0.7)Ω

(v) Error in the power of a quantity

Consider the n^{th} power of A, $Z = A^n$

The error ΔZ in Z is given by

$$Z \pm \Delta Z = (A \pm \Delta A)^n = A^n\left(1 \pm \frac{\Delta A}{A}\right)^n$$

$$= Z\left(1 \pm n\frac{\Delta A}{A}\right)$$

We get $[(1+x)^n \approx 1+nx$, when $x<<1]$ neglecting remaining terms, Dividing both sides by Z

$$1 \pm \frac{\Delta Z}{Z} = 1 \pm n\frac{\Delta A}{A} \text{ or}$$

$$\frac{\Delta Z}{Z} = n.\frac{\Delta A}{A} \qquad (1.7)$$

The fractional error in the n^{th} power of a quantity is n times the fractional error in that quantity.

General rule: If $Z = \frac{A^p B^q}{C^r}$ Then

maximum fractional error in Z is

given by $\frac{\Delta Z}{Z} = p\frac{\Delta A}{A} + q\frac{\Delta B}{B} + r\frac{\Delta C}{C}$

The percentage error in Z is given by

$$\frac{\Delta Z}{Z} \times 100 = p\frac{\Delta A}{A} \times 100 + q\frac{\Delta B}{B} \times 100$$
$$+ r\frac{\Delta C}{C} \times 100$$

EXAMPLE 1.9

A physical quantity x is given by $x = \frac{a^2 b^3}{c\sqrt{d}}$. If the percentage errors of measurement in a, b, c and d are 4%, 2%, 3% and 1% respectively, then calculate the percentage error in the calculation of x. (NEET 2013)

Solution

Given $x = \frac{a^2 b^3}{c\sqrt{d}}$

The percentage error in x is given by

$$\frac{\Delta x}{x}\times 100 = 2\frac{\Delta a}{a}\times 100 \;+3\frac{\Delta b}{b}\times 100$$

$$+\frac{\Delta c}{c}\times 100+\frac{1}{2}\frac{\Delta d}{d}\times 100$$

$= (2 \times 4\%) + (3 \times 2\%) + (1 \times 3\%) + (½ \times 1\%)$

$= 8\% +6\%+3\% +0.5\%$

The percentage error is $x = 17.5\%$

1.7 SIGNIFICANT FIGURES

1.7.1 Definition and Rules of Significant Figures

Suppose we ask three students to measure the length of a stick using metre scale (the least count for metre scale is 1 mm or 0.1 cm). So, the result of the measurement (length of stick) can be any of the following, 7.20 cm or 7.22 cm or 7.23 cm. Note that all the three students measured first two digits correctly (with confidence) but last digit varies from person to person. So, the number of meaningful digits is 3 which communicate both measurement (quantitative) and also the precision of the instrument used. Therefore, significant number or significant digit is 3. It is defined as the **number of meaningful digits which contain numbers that are known reliably and first uncertain number**.

Examples: The significant figure for the digit 121.23 is 5, significant figure for the digit 1.2 is 2, significant figure for the digit 0.123 is 3, significant digit for 0.1230 is 4, significant digit for 0.0123 is 3, significant digit for 1230 is 3, significant digit for 1230 (with decimal) is 4 and significant digit for 20000000 is 1 (because $20000000=2 \times 10^7$ has only one significant digit, that is, 2).

In physical measurement, if the length of an object is $l = 1230$ m, then significant digit for l is 4.

The rules for counting significant figures are given in Table 1.9.

EXAMPLE 1.10

State the number of significant figures in the following

i) 600800
ii) 400
iii) 0.007
iv) 5213.0
v) $2.65 \times 10^{24}\, m$
vi) 0.0006032

Solution: i) four ii) one iii) one iv) five v) three vi) four

1.7.2 Rounding Off

Calculators are widely used now-a-days to do calculations. The result given by a calculator has too many figures. In no case should the result have more significant figures than the figures involved in the data used for calculation. The result of calculation with numbers containing more than one uncertain digit should be rounded off. The rules for rounding off are shown in Table 1.10.

EXAMPLE 1.11

Round off the following numbers as indicated

i) 18.35 up to 3 digits
ii) 19.45 up to 3 digits
iii) 101.55×10^6 up to 4 digits
iv) 248337 up to digits 3 digits
v) 12.653 up to 3 digits.

Table 1.9 Rules for counting significant figures

	Rule	Example
i)	All non-zero digits are significant	1342 has **four** significant figures
ii)	All zeros between two non zero digits are significant	2008 has **four** significant figures
iii)	All zeros to the right of a non-zero digit but to the left of a decimal point are significant.	30700. has **five** significant figures
iv)	For the number without a decimal point, the terminal or trailing zero(s) are not significant.	30700 has **three** significant figures
v)	If the number is less than 1, the zero (s) on the right of the decimal point but to left of the first non zero digit are not significant.	0.00345 has **three** significant figures
vi)	All zeros to the right of a decimal point and to the right of non-zero digit are significant.	40.00 has **four** significant figures and 0.030400 has **five** significant figures
vii)	The number of significant figures does not depend on the system of units used	1.53 cm, 0.0153 m, 0.0000153 km, all have **three** significant figures

Note 1: Multiplying or dividing factors, which are neither rounded numbers nor numbers representing measured values, are exact and they have infinite numbers of significant figures as per the situation. For example, circumference of circle S = 2πr, Here the factor 2 is exact number. It can be written as 2.0, 2.00 or 2.000 as required.

Note 2: The power of 10 is irrelevant to the determination of significant figures.
For example $x = 5.70 \text{ m} = 5.70 \times 10^2 \text{ cm} = 5.70 \times 10^3 \text{ mm} = 5.70 \times 10^{-3} \text{ km}$.
In each case the number of significant figures is **three.**

Table 1.10 Rules for Rounding Off

Rule	Example
i) If the digit to be dropped is smaller than 5, then the preceding digit should be left unchanged.	i) 7.32 is rounded off to 7.3 ii) 8.94 is rounded off to 8.9
ii) If the digit to be dropped is greater than 5, then the preceding digit should be increased by 1	i) 17.26 is rounded off to 17.3 ii) 11.89 is rounded off to 11.9
iii) If the digit to be dropped is 5 followed by digits other than zero, then the preceding digit should be raised by 1	i) 7.352, on being rounded off to first decimal becomes 7.4 ii) 18.159 on being rounded off to first decimal, become 18.2
iv) If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit is not changed if it is even	i) 3.45 is rounded off to 3.4 ii) 8.250 is rounded off to 8.2
v) If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit is raised by 1 if it is odd	i) 3.35 is rounded off to 3.4 ii) 8.350 is rounded off to 8.4

Solution

i) 18.4 ii) 19.4 iii) 101.6×10^6
iv) 248000 v) 12.7

1.7.3 Arithmetical Operations with Significant Figures

(i) Addition and subtraction

In addition and subtraction, the final result should retain as many decimal places as there are in the number with the smallest number of decimal places.

Example:

1. $3.1 + 1.780 + 2.046 = 6.926$

   Here the least number of significant digits after the decimal is one. Hence the result will be 6.9.

2. $12.637 - 2.42 = 10.217$

   Here the least number of significant digits after the decimal is two. Hence the result will be 10.22

(ii) Multiplication and Division

In multiplication or division, the final result should retain as many significant figures as there are in the original number with smallest number of significant figures.

Example:

1. $1.21 \times 36.72 = 44.4312 = 44.4$

   Here the least number of significant digits in the measured values is three. Hence the result when rounded off to three significant digits is 44.4

2. $36.72 \div 1.2 = 30.6 = 31$

   Here the least number of significant digits in the measured values is two. Hence the result when rounded off to significant digit becomes 31.

1.8 DIMENSIONAL ANALYSIS

1.8.1 Dimension of Physical Quantities

In mechanics, we deal with the physical quantities like mass, time, length, velocity, acceleration, etc. which can be expressed in terms of three independent base quantities such as M, L and T. So, the dimension of a physical quantity can be defined as 'any physical quantity which is expressed in terms of base quantities whose exponent (power) represents the dimension of the physical quantity'. The notation used to denote the dimension of a physical quantity is [(physical quantity within square bracket)]. For an example, [length] means dimension of length, [area] means dimension of area, etc. The dimension of length can be expressed in terms of base quantities as

$$[\text{length}] = M^0 L T^0 = L$$

Similarly, $[\text{area}] = M^0 L^2 T^0 = L^2$

Similarly, $[\text{volume}] = M^0 L^3 T^0 = L^3$

Note that in all the cases, the base quantity L is same but exponent (power) are different, which means dimensions are different. For a pure number, exponent of base quantity is zero. For example, consider the number 2, which has no dimension and can be expressed as

$$\Rightarrow [2] = M^0 L^0 T^0 \qquad \text{(dimensionless)}$$

Let us write down the dimensions of a few more physical quantities.

$$\text{Speed}, s = \frac{\text{distance}}{\text{time taken}} \Rightarrow [s] = \frac{L}{T} = LT^{-1}$$

$$\text{Velocity}, \vec{v} = \frac{\text{displacement}}{\text{time taken}} \Rightarrow [\vec{v}] = \frac{L}{T} = LT^{-1}$$

Note that speed is a scalar quantity and velocity is a vector quantity (scalar and vector will be discussed in Unit 2) but both of them have the same dimensional formula.

$$\text{Acceleration}, \vec{a} = \frac{\text{velocity}}{\text{time taken}} \Rightarrow [\vec{a}] = \frac{\mathrm{LT}^{-1}}{\mathrm{T}} = \mathrm{LT}^{-2}$$

Acceleration is velocity per time.
Linear momentum or Momentum,

$$[\vec{p}] = m\vec{v} \Rightarrow [\vec{p}] = MLT^{-1}$$

$$\text{Force}, \vec{F} = m\vec{a} \Rightarrow [\vec{F}] = MLT^{-2} = \frac{\text{Momentum}}{\text{time}}$$

This is true for any kind of force. There are only four types of forces that exist in nature viz strong force, electromagnetic force, weak force and gravitational force. Further, frictional force, centripetal force, centrifugal force, all have the dimension MLT^{-2}.

$$\text{Impulse}, \vec{I} = \vec{F}t \Rightarrow [\vec{I}] = MLT^{-1}$$

$$= \text{dimension of momentum}$$

Angular momentum is the moment of linear momentum (discussed in unit 5).

$$\text{Angular Momentum}, \vec{\mathrm{L}} = \vec{r} \times \vec{p} \Rightarrow [\vec{\mathrm{L}}] = ML^2T^{-1}$$

Work done,

$$W = \vec{F} \cdot \vec{d} \Rightarrow [W] = ML^2T^{-2}$$

Kinetic energy

$$KE = \frac{1}{2}mv^2 \Rightarrow [KE] = \left[\frac{1}{2}\right][m][v^2]$$

Since, number $\frac{1}{2}$ is dimensionless, the dimension of kinetic energy $[KE] = [m][v^2] = ML^2T^{-2}$. Similarly, to get the dimension of potential energy, let us consider the gravitational potential energy, $PE = mgh \Rightarrow [PE] = [m][g][h]$, , where, m is the mass of the particle, g is the acceleration due to gravity and h is the height from the ground level. Hence, $[PE] = [m][g][h] = ML^2T^{-2}$. Thus, for any kind of energy (such as for internal energy, total energy etc), the dimension is

$$[Energy] = ML^2T^{-2}$$

The moment of force is known as torque, $\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow [\vec{\tau}] = ML^2T^{-2}$ (Read the symbol τ as tau – Greek alphabet). Note that the dimension of torque and dimension of energy are identical but they are different physical quantities. Further one of them is a scalar (energy) and another one is a vector (torque). This means that the dimensionally same physical quantities need not be the same physical quantities.

Note

1. We may come across dimensions in different situations in physics, so we often confuse with the term 'dimension'. For instance, we come across terms like 'dimension of energy', 'motion in one dimension' and 'dimension of atom'. It should be kept in mind that dimension of physical quantity means expressing physical quantity in terms of exponent of the base quantity. Motion in one dimension, two dimensions and three dimensions implies that it gives dimension of space. Dimension of atom implies the size of the atom. So, simply writing dimension is meaningless. Hence, the meaning should be taken with the context we write.

2. All the trigonometric functions like sinθ, cosθ etc. are dimensionless (θ is dimensionless), exponential function e^x and logarithm function *ln x* are dimensionless (x must be dimensionless). Suppose we expand a function in series expansion (finite or infinite) which contain terms like, $x^0, x^1, x^2, \ldots$ then x must be dimensionless quantity.

Table 1.11 Dimensional Formula

Physical quantity	Expression	Dimensional formula
Area (Rectangle)	length × breadth	$[L^2]$
Volume	Area × height	$[L^3]$
Density	mass / volume	$[ML^{-3}]$
Velocity	displacement/time	$[LT^{-1}]$
Acceleration	velocity / time	$[LT^{-2}]$
Momentum	mass × velocity	$[MLT^{-1}]$
Force	mass × acceleration	$[MLT^{-2}]$
Work	force × distance	$[ML^2T^{-2}]$
Power	work / time	$[ML^2T^{-3}]$
Energy	Work	$[ML^2T^{-2}]$
Impulse	force × time	$[MLT^{-1}]$
Radius of gyration	Distance	$[L]$
Pressure (or) stress	force / area	$[ML^{-1}T^{-2}]$
Surface tension	force / length	$[MT^{-2}]$
Frequency	1 / time period	$[T^{-1}]$
Moment of Inertia	mass × (distance)^2	$[ML^2]$
Moment of force (or torque)	force × distance	$[ML^2T^{-2}]$
Angular velocity	angular displacement / time	$[T^{-1}]$
Angular acceleration	angular velocity / time	$[T^{-2}]$
Angular momentum	linear momentum × distance	$[ML^2T^{-1}]$
Co-efficient of Elasticity	stress/strain	$[ML^{-1}T^{-2}]$
Co-efficient of viscosity	(force × distance) / (area × velocity)	$[ML^{-1}T^{-1}]$
Surface energy	work / area	$[MT^{-2}]$
Heat capacity	heat energy / temperature	$[ML^2T^{-2}K^{-1}]$
Charge	current × time	$[AT]$
Magnetic induction	force / (current × length)	$[MT^{-2}A^{-1}]$
Force constant	force / displacement	$[MT^{-2}]$
Gravitational constant	[force × (distance)^2] / (mass)^2	$[M^{-1}L^3T^{-2}]$
Planck's constant	energy / frequency	$[ML^2T^{-1}]$
Faraday constant	avogadro constant × elementary charge	$[AT\ mol^{-1}]$
Boltzmann constant	energy / temperature	$[ML^2T^{-2}K^{-1}]$

1.8.2 Dimensional Quantities, Dimensionless Quantities, Principle of Homogeneity

On the basis of dimension, we can classify quantities into four categories.

(1) **Dimensional variables**

Physical quantities, which possess dimensions and have variable values are called dimensional variables. Examples are length, velocity, and acceleration etc.

(2) **Dimensionless variables**

Physical quantities which have no dimensions, but have variable values are called dimensionless variables. Examples are specific gravity, strain, refractive index etc.

(3) **Dimensional Constant**

Physical quantities which possess dimensions and have constant values are called dimensional constants. Examples are Gravitational constant, Planck's constant etc.

(4) **Dimensionless Constant**

Quantities which have constant values and also have no dimensions are called dimensionless constants. Examples are π, e (Euler's number), numbers etc.

Principle of homogeneity of dimensions

The principle of homogeneity of dimensions states that the dimensions of all the terms in a physical expression should be the same. For example, in the physical expression $v^2 = u^2 + 2as$, the dimensions of v^2, u^2 and 2 as are the same and equal to $[L^2T^{-2}]$.

1.8.3 Application and Limitations of the Method of Dimensional Analysis.

This method is used to

(i) Convert a physical quantity from one system of units to another.

(ii) Check the dimensional correctness of a given physical equation.

(iii) Establish relations among various physical quantities.

(i) **To convert a physical quantity from one system of units to another**

This is based on the fact that the product of the numerical values (n) and its corresponding unit (u) is a constant. i.e, n [u] = constant (or) $n_1[u_1] = n_2[u_2]$.

Consider a physical quantity which has dimension '*a*' in mass, '*b*' in length and '*c*' in time. If the fundamental units in one system are M_1, L_1 and T_1 and the other system are M_2, L_2 and T_2 respectively, then we can write, $n_1\ [M_1^a L_1^b T_1^c] = n_2\ [M_2^a L_2^b T_2^c]$

We have thus converted the numerical value of physical quantity from one system of units into the other system.

EXAMPLE 1.12

Convert 76 cm of mercury pressure into Nm^{-2} using the method of dimensions.

Solution

In cgs system 76 cm of mercury pressure = $76 \times 13.6 \times 980$ dyne cm^{-2}

The dimensional formula of pressure P is $[ML^{-1}T^{-2}]$

$$P_1\left[M_1^a L_1^b T_1^c\right] = P_2\left[M_2^a L_2^b T_2^c\right]$$

We have $P_2 = P_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$

$M_1 = 1g$, $M_2 = 1kg$
$L_1 = 1$ cm, $L_2 = 1m$
$T_1 = 1$ s, $T_2 = 1s$

As a =1, b= -1, and c = -2

Then

$$P_2 = 76\times13.6\times980\left[\frac{1g}{1kg}\right]^1\left[\frac{1cm}{1m}\right]^{-1}\left[\frac{1s}{1s}\right]^{-2}$$

$$=76\times13.6\times980\left[\frac{10^{-3}kg}{1kg}\right]^1\left[\frac{10^{-2}m}{1m}\right]^{-1}\left[\frac{1s}{1s}\right]^{-2}$$

$$= 76 \times 13.6 \times 980 \times [10^{-3}] \times 10^2$$

$$P_2 = 1.01 \times 10^5 \text{ Nm}^{-2}$$

EXAMPLE 1.13

If the value of universal gravitational constant in SI is $6.6\text{x}10^{-11}$ $\text{Nm}^2\,\text{kg}^{-2}$, then find its value in CGS System?

Solution

Let G_{SI} be the gravitational constant in the SI system and G_{cgs} in the cgs system. Then

$$G_{SI} = 6.6\times10^{-11}\text{Nm}^2\,\text{kg}^{-2}$$

$$G_{cgs} = ?$$

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

$$G_{cgs} = G_{SI}\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

$M_1 = 1$ kg $\quad L_1 = 1$m $\quad T_1 = 1$s

$M_2 = 1$ g $\quad L_2 = 1$ cm $\quad T_2 = 1$s

The dimensional formula for G is $M^{-1}L^3T^{-2}$

a = -1 b = 3 and c = -2

$$G_{cgs} = 6.6\times10^{-11}\left[\frac{1kg}{1g}\right]^{-1}\left[\frac{1m}{1cm}\right]^3\left[\frac{1s}{1s}\right]^{-2}$$

$$= 6.6\times10^{-11}\left[\frac{1kg}{10^{-3}kg}\right]^{-1}\left[\frac{1m}{10^{-2}m}\right]^3\left[\frac{1s}{1s}\right]^{-2}$$

$$= 6.6 \times 10^{-11} \times 10^{-3}\text{x } 10^6 \times 1$$

$$G_{cgs} = 6.6 \times 10^{-8}\,dyne\ \text{cm}^2\,g^{-2}$$

(ii) To check the dimensional correctness of a given physical equation

Let us take the equation of motion v = u + at

Apply dimensional formula on both sides

$[LT^{-1}] = [LT^{-1}] + [LT^{-2}]\,[T]$

$$[LT^{-1}] = [LT^{-1}] + [LT^{-1}]$$

(Quantities of same dimension only can be added)

We see that the dimensions of both sides are same. Hence the equation is dimensionally correct.

EXAMPLE 1.14

Check the correctness of the equation $\frac{1}{2}mv^2 = mgh$ using dimensional analysis method.

Solution

Dimensional formula for

$$\frac{1}{2}mv^2 = [M][LT^{-1}]^2 = [ML^2T^{-2}]$$

Dimensional formula for

$$mgh = [M][LT^{-2}][L] = [ML^2T^{-2}]$$

$$[ML^2T^{-2}] = [ML^2T^{-2}]$$

Both sides are dimensionally the same, hence the equations $\frac{1}{2}mv^2 = mgh$ is dimensionally correct.

(iii) To establish the relation among various physical quantities

If the physical quantity Q depends upon the quantities Q_1, Q_2 and Q_3 ie. Q is proportional to Q_1, Q_2 and Q_3.

Then,

$$Q \propto Q_1^a Q_2^b Q_3^c$$

$$Q = k Q_1^a Q_2^b Q_3^c$$

where k is a dimensionless constant. When the dimensional formula of Q, Q_1, Q_2 and Q_3 are substituted, then according to the principle of homogeneity, the powers of M, L, T are made equal on both sides of the equation. From this, we get the values of a, b, c

EXAMPLE 1.15

Obtain an expression for the time period T of a simple pendulum. The time period T depends on (i) mass '*m*' of the bob (ii) length '*l*' of the pendulum and (iii) acceleration due to gravity *g* at the place where the pendulum is suspended. (Constant k = 2π) i.e

Solution

$$T \propto m^a l^b g^c$$

$$T = km^a l^b g^c$$

Here k is the dimensionless constant. Rewriting the above equation with dimensions

$$[T^1] = [M^a]\,[L^b]\,[LT^{-2}]^c$$

$$[M^0L^0T^1] = [M^a\,L^{b+c}\,T^{-2c}]$$

Comparing the powers of M, L and T on both sides, a=0, b+c=0, -2c=1

Solving for a,b and c a = 0, b = 1/2, and c = −1/2

From the above equation T = km^0 $\ell^{1/2}$ g$^{-1/2}$

$$T = k\left(\frac{\ell}{g}\right)^{1/2} = k\sqrt{\ell/g}$$

Experimentally k = 2π, hence $T = 2\pi\sqrt{\ell/g}$

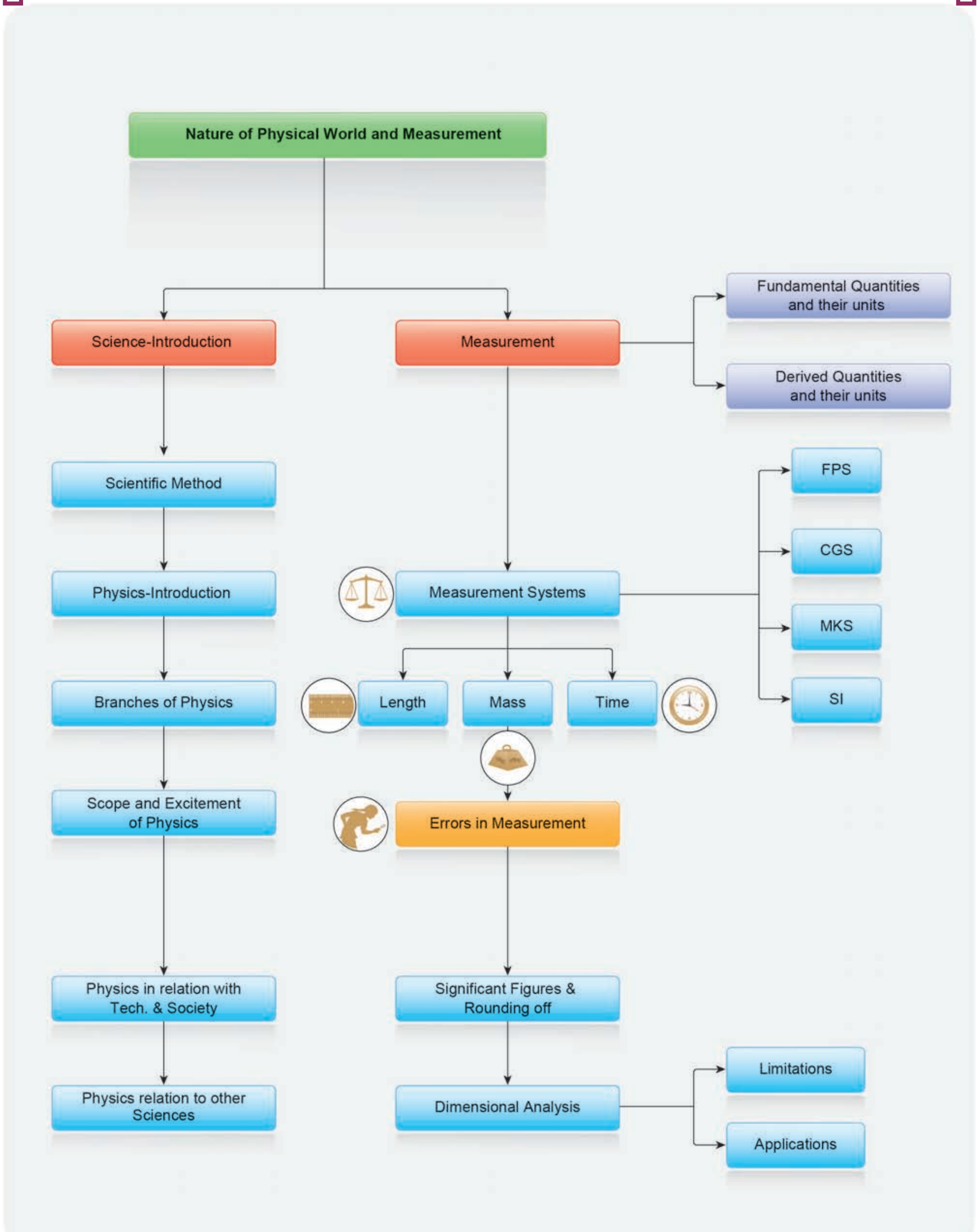
Limitations of Dimensional analysis

1. This method gives no information about the dimensionless constants in the formula like 1, 2, ……..π, e (Euler number), etc.
2. This method cannot decide whether the given quantity is a vector or a scalar.
3. This method is not suitable to derive relations involving trigonometric, exponential and logarithmic functions.
4. It cannot be applied to an equation involving more than three physical quantities.
5. It can only check on whether a physical relation is dimensionally correct but not the correctness of the relation. For example using dimensional analysis, $s = ut + \frac{1}{3}at^2$ is dimensionally correct whereas the correct relation is $s = ut + \frac{1}{2}at^2$.

SUMMARY

- Physics is an experimental science in which measurements made must be expressed in units.
- All physical quantities have a magnitude (size) and a unit.
- The SI unit of length, mass, time, temperature, electric current, amount of substance and luminous intensity are metre, kilogram, second, kelvin, ampere, mole and candela respectively.
- Units of all mechanical, electrical, magnetic and thermal quantities are derived in terms of these base units.
- Screw gauge, vernier caliper methods are available for the measurement of length in the case of small distances.
- Parallax, RADAR methods are available for the measurement of length in the case of long distances.
- The uncertainty in a measurement is called error. The accuracy of a measurement is a measure of how close the measured value is to the true value of the quantity. Every accurate measurement is precise but every precise measurement need not be accurate.
- When two or more quantities are added or subtracted, the result can be as precise as the least of the individual precisions. When the quantities are multiplied or divided, the result has the same number of significant figures as the quantity with the smallest number of significant figures.
- Dimensional analysis is used to perform quick check on the validity of equations. Whenever the quantities are added, subtracted or equated, they must have the same dimension. A dimensionally correct equation may not be a true equation but every true equation is necessarily dimensionally correct.

CONCEPT MAP



Nature of Physical World and Measurement
Science-Introduction
Measurement
Fundamental Quantities and their units
Derived Quantities and their units
Scientific Method
Physics-Introduction
Branches of Physics
Scope and Excitement of Physics
Physics in relation with Tech. & Society
Physics relation to other Sciences
Measurement Systems
FPS
CGS
MKS
SI
Length
Mass
Time
Errors in Measurement
Significant Figures & Rounding off
Dimensional Analysis
Limitations
Applications

EVALUATION

I. Multiple Choice Questions

1. One of the combinations from the fundamental physical constants is $\frac{hc}{G}$. The unit of this expression is

a) kg^2 b) m^3

c) s^{-1} d) m

2. If the error in the measurement of radius is 2%, then the error in the determination of volume of the sphere will be

a) 8% b) 2%

c) 4% d) 6%

3. If the length and time period of an oscillating pendulum have errors of 1% and 3% respectively then the error in measurement of acceleration due to gravity is

a) 4%

b) 5%

c) 6%

d) 7%

4. The length of a body is measured as 3.51 m, if the accuracy is 0.01m, then the percentage error in the measurement is

a) 351% b) 1%

c) 0.28% d) 0.035%

5. Which of the following has the highest number of significant figures?

a) 0.007 m^2

b) 2.64×10^{24} kg

c) 0.0006032 m^2

d) 6.3200 J

6. If $\pi = 3.14$, then the value of π^2 is

a) 9.8596

b) 9.860

c) 9.86

d) 9.9

7. Round of the following number 19.95 into three significant figures.

a) 19.9 b) 20.0

c) 20.1 d) 19.5

8. Which of the following pairs of physical quantities have same dimension?

a) force and power

b) torque and energy

c) torque and power

d) force and torque

9. The dimensional formula of Planck's constant h is [JEE Main, NEET]

a) $[ML^2T^{-1}]$ b) $[ML^2T^{-3}]$

c) $[MLT^{-1}]$ d) $[ML^3T^{-3}]$

10. The velocity of a particle v at an instant t is given by $v = at + bt^2$. The dimensions of b is

a) [L] b) $[LT^{-1}]$

c) $[LT^{-2}]$ d) $[LT^{-3}]$

11. The dimensional formula for gravitational constant G is

a) $[ML^3T^{-2}]$ b) $[M^{-1}L^3T^{-2}]$

c) $[M^{-1}L^{-3}T^{-2}]$ d) $[ML^{-3}T^2]$

12. The density of a material in CGS system of units is 4 g cm^{-3} . In a system of units in which unit of length is 10 cm and unit of mass is 100 g, then the value of density of material will be

a) 0.04 b) 0.4

c) 40 d) 400

13. If the force is proportional to square of velocity, then the dimension of proportionality constant is [JEE-2000]

a) [MLT^0] b) [MLT^{-1}]

c) [ML^{-2}T] d) [ML^{-1}T^0]

14. The dimension of $(\mu_0\varepsilon_0)^{-\frac{1}{2}}$ is [Main AIPMT 2011]

(a) length (b) time

(c) velocity (d) force

15. Planck's constant (h), speed of light in vacuum (c) and Newton's gravitational constant (G) are taken as three fundamental constants. Which of the following combinations of these has the dimension of length? [NEET 2016 (phase II)]

(a) $\frac{\sqrt{hG}}{c^{\frac{3}{2}}}$ (b) $\frac{\sqrt{hG}}{c^{\frac{5}{2}}}$

(c) $\sqrt{\frac{hc}{G}}$ (d) $\sqrt{\frac{Gc}{h^{\frac{3}{2}}}}$

Answers:

1) a	2) d	3) d	4) c
5) d	6) c	7) b	8) b
9) a	10) d	11) b	12) c
13) d	14) c	15) a	

II. Short Answer Questions

1. Briefly explain the types of physical quantities.

2. How will you measure the diameter of the Moon using parallax method?

3. Write the rules for determining significant figures.

4. What are the limitations of dimensional analysis?

5. Define precision and accuracy. Explain with one example.

III. Long Answer Questions

1. i) Explain the use of screw gauge and vernier caliper in measuring smaller distances.

ii) Write a note on triangulation method and radar method to measure larger distances.

2. Explain in detail the various types of errors.

3. What do you mean by propagation of errors? Explain the propagation of errors in addition and multiplication.

4. Write short notes on the following.

a) Unit

b) Rounding - off

c) Dimensionless quantities

5. Explain the principle of homogeneity of dimensions. Give example.

IV. Exercises

1. In a submarine equipped with sonar, the time delay between the generation of a pulse and its echo after reflection from an enemy submarine is observed to be 80 s. If the speed of sound in water is 1460 ms^{-1}. What is the distance of enemy submarine? Ans: (58.40 km)

2. The radius of the circle is 3.12 m. Calculate the area of the circle with regard to significant figures. Ans: (30.6 m^2)

3. Assuming that the frequency γ of a vibrating string may depend upon i) applied force (F) ii) length (ℓ) iii) mass per unit length (m), prove that $\gamma \alpha \frac{1}{l}\sqrt{\frac{F}{m}}$ using dimensional analysis. (related to JIPMER 2001)

4. Jupiter is at a distance of 824.7 million km from the Earth. Its angular diameter is measured to be 35.72″. Calculate the diameter of Jupiter. Ans: (1.428×10^5 km)

5. The measurement value of length of a simple pendulum is 20 cm known with 2 mm accuracy. The time for 50 oscillations was measured to be 40 s within 1 s resolution. Calculate the percentage accuracy in the determination of acceleration due to gravity 'g' from the above measurement. Ans: (6%)

BOOKS FOR REFERENCE

1. Karen Cummings, Priscilla Laws, Edward Redish, Patrick Cooney, Understanding Physics, Wiley India Pvt LTd, 2nd Edition 2007.
2. Sears and Zemansky's, College Physics, Pearson Educatoin Ltd,10th Edition, 2016.
3. Halliday. D and Resnick.R, Physics. Part-I, Wiley Easter, New Delhi
4. Sanjay Moreshwar Wagh and Dilip Abasaheb Deshpande, Essentials of Physics Volume I, PHI learning Pvt Ltd, New Delhi. 2013.
5. James S. Walker , Physics, Addison-Wesley Publishers, 4th Edition

ICT CORNER

Screw Gauge and Vernier Caliper

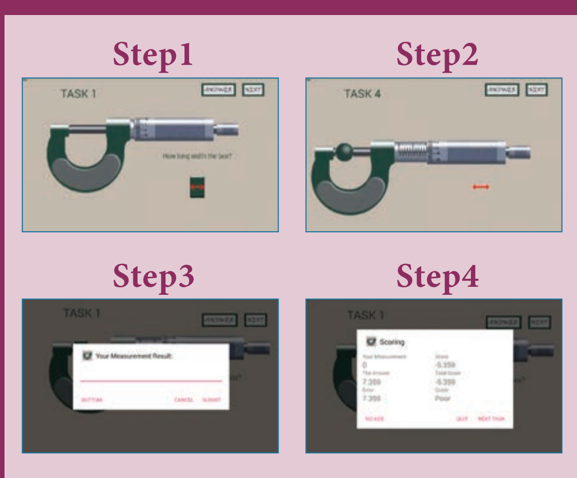
Measure with Pleasure

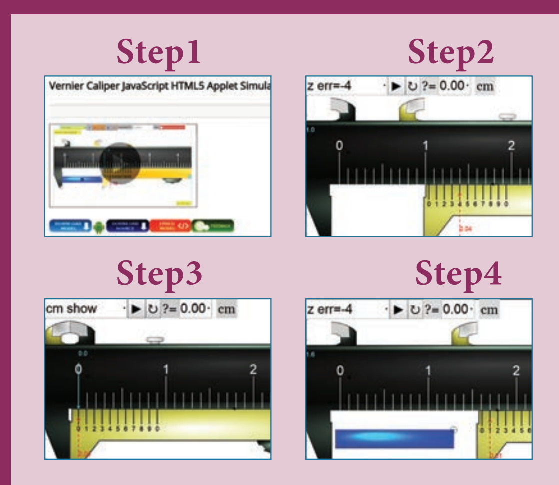
STEPS:

- Get into/Access the application with the help of the link given below or the given QR code.
- To measure the given object's diameter / thickness, place the object as it should be fitted between the screw and anvil. The object can be fitted properly by adjusting the screw.
- After clicking the Answer button, You will get a box showing Your Measurement Result. Input your measured value in it and click Submit button. There you can verify your measurement if it is right or wrong.
- You can measure the diameter of various objects by clicking Next button.

STEPS:

- Use the given URL to access 'Vernier Caliper' simulation page. Click 'Play Button' to launch the simulation.
- Select the unit and set the 'Zero Error' from the dropdown above the scale.
- Click and drag the secondary scale and place the blue coloured object in between them. Find the measurement and enter it in the answer box above the scale.
- Change the size of the object by clicking and dragging the edge of the blue object and practice the measurement using Vernier Caliper.





Screw Gauge stimulation URL:

https://play.google.com/store/apps/details?id=com.priantos.screwgaugegames&hl=en

Vernier caliper stimulation URL:

http://iwant2study.org/ospsg/index.php/interactive-resources/physics/01-measurements/5-vernier-caliper#faqnoanchor

* Pictures are indicative only.

* If browser requires, allow **Flash Player** or **Java Script** to load the page.

UNIT 2 KINEMATICS

All the laws of nature are written in the language of mathematics– Galileo

Learning Objectives

In this unit, the student is exposed to

- different types of motions (linear, rotational and oscillatory)
- the necessity for reference frames to explain the motion of objects
- the meaning of vectors, scalars and their properties
- the role of scalar and vector products in physics
- the basics of differentiation and integration
- the notions of displacement and distance and their variation with time
- the notions of speed, velocity, acceleration and their graphs.
- the notion of relative velocity
- kinematic equations of motion for constant acceleration
- analysis of various types of motion of objects under gravitational force
- radians and degrees
- uniform circular motion, centripetal acceleration and centripetal force.

2.1 INTRODUCTION

Physics is basically an experimental science and rests on two pillars–Experiments and mathematics. Two thousand three hundred years ago the Greek librarian Eratosthenes measured the radius of the Earth. The size of the atom was measured much later, only in the beginning of the 20th century. The central aspect in physics is motion. Motion is found at all levels–from microscopic level (within the atom) to macroscopic and galactic level (planetary system and beyond). In short the entire Universe is governed by various types of motion. The study of various types of motion is expressed using the language of mathematics.

How do objects move? How fast or slow do they move? For example, when ten athletes run in a race, all of them do not run in the same manner. Their performance cannot be qualitatively recorded by usage of words like 'fastest', 'faster', 'average', 'slower' or 'slowest'. It has to be quantified. Quantifying means assigning numbers to each athlete's motion. Comparing these numbers, one can analyse how fast or slow each athlete runs when compared to others. In this unit, the basic mathematics needed for analyzing motion in terms of its direction and magnitude is covered.

Kinematics is the branch of mechanics which deals with the motion of objects without taking force into account. The Greek word ***"kinema"*** means ***"motion"***.

2.2 CONCEPT OF REST AND MOTION

The concept of rest and motion can be well understood by the following elucidation (Figure 2.1). A person sitting in a moving bus is at rest with respect to a fellow passenger but is in motion with respect to a person outside the bus. The concepts of rest and motion have meaning only with respect to some reference frame. To understand rest or motion we need a convenient fixed reference frame.

Figure 2.1 Frame of Reference

Frame of Reference:

If we imagine a coordinate system and the position of an object is described relative to it, then such a coordinate system is called frame of reference.

At any given instant of time, the frame of reference with respect to which the position of the object is described in terms of position coordinates (x, y, z) (i.e., distances of the given position of an object along the x, y, and z–axes.) is called **"Cartesian coordinate system"** as shown in Figure 2.2

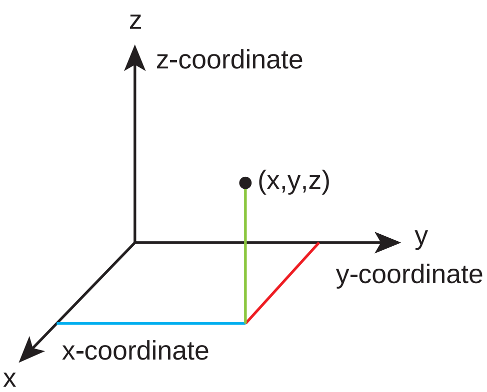


Figure 2.2 Cartesian coordinate system

It is to be noted that if the *x, y* and *z* axes are drawn in anticlockwise direction then the coordinate system is called as "right–handed Cartesian coordinate system". Though other coordinate systems do exist, in physics we conventionally follow the right–handed coordinate system as shown in Figure 2.3.

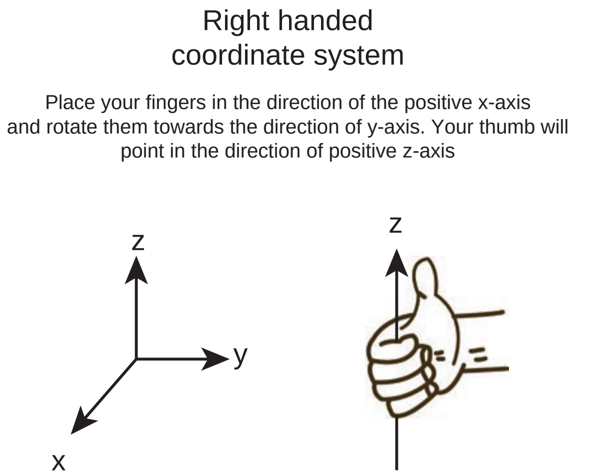


Figure 2.3 Right handed coordinate system

The following Figure 2.4 illustrates the difference between left and right handed coordinate systems.

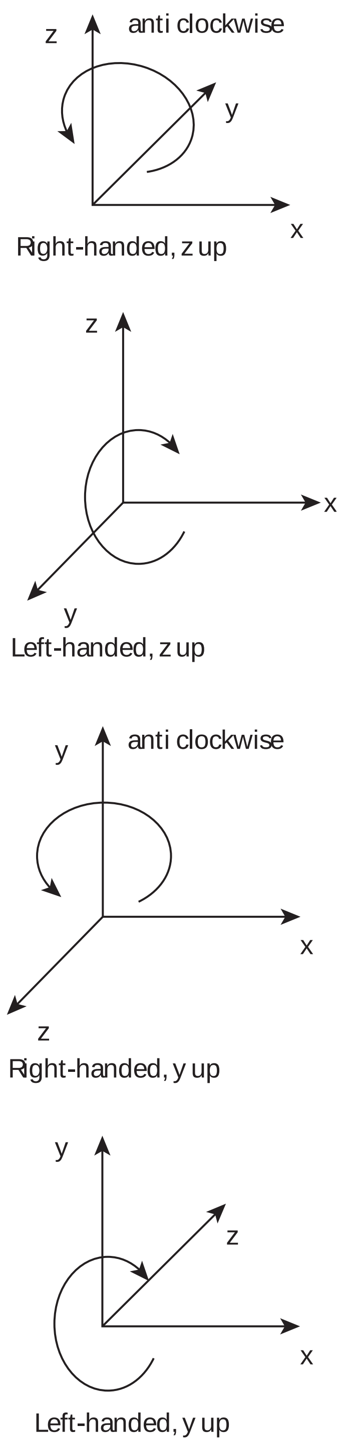


Figure 2.4 Right and left handed coordinate systems

Point mass

To explain the motion of an object which has finite mass, the concept of "point mass" is required and is very useful. Let the mass of any object be assumed to be concentrated at a point. Then this idealized mass is called "point mass". It has no internal structure like shape and size. Mathematically a point mass has finite mass with zero dimension. Even though in reality a point mass does not exist, it often simplifies our calculations. It is to be noted that the term "point mass" is a relative term. It has meaning only with respect to a reference frame and with respect to the kind of motion that we analyse.

Examples

- To analyse the motion of Earth with respect to Sun, Earth can be treated as a point mass. This is because the distance between the Sun and Earth is very large compared to the size of the Earth.
- If we throw an irregular object like a small stone in the air, to analyse its motion it is simpler to consider the stone as a point mass as it moves in space. The size of the stone is very much smaller than the distance through which it travels.

Types of motion

In our day-to-day life the following kinds of motion are observed:

a) Linear motion

An object is said to be in linear motion if it moves in a straight line.

Examples

- An athlete running on a straight track
- A particle falling vertically downwards to the Earth.

b) Circular motion

Circular motion is defined as a motion described by an object traversing a circular path.

Examples

- The whirling motion of a stone attached to a string

- The motion of a satellite around the Earth

These two circular motions are shown in Figure 2.5

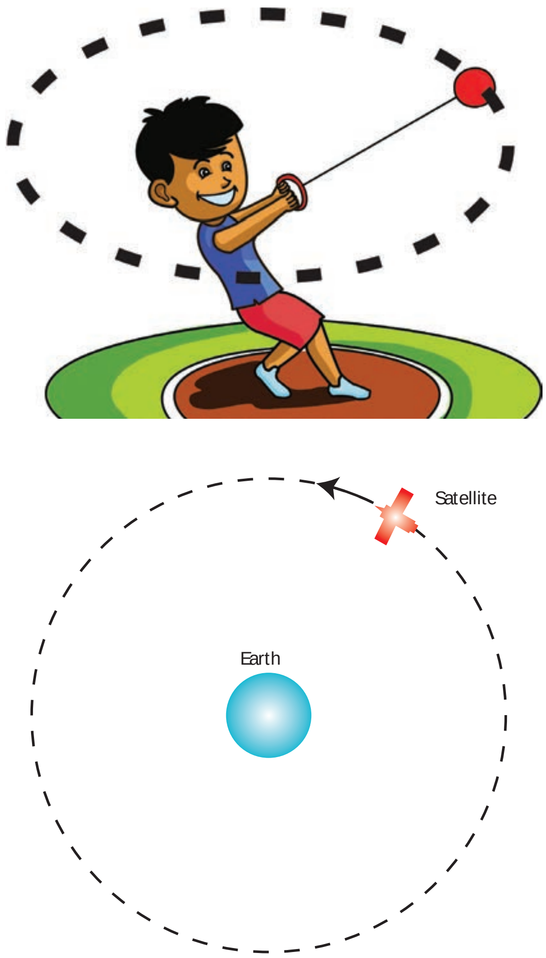


Figure 2.5 Examples of circular motion

c) Rotational motion

If any object moves in a rotational motion about an axis, the motion is called 'rotation'. During rotation every point in the object transverses a circular path about an axis, (except the points located on the axis).

Examples

- Rotation of a disc about an axis through its centre
- Spinning of the Earth about its own axis.

These two rotational motions are shown in Figure 2.6.

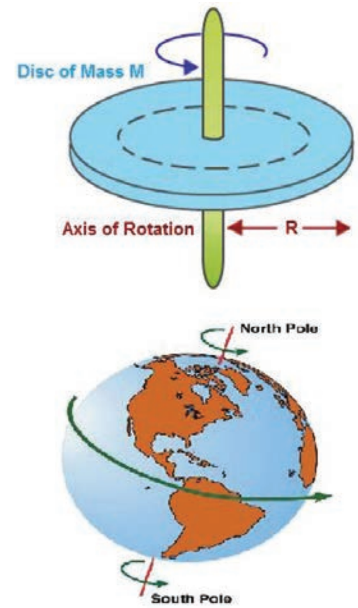


Figure 2.6 Examples of Rotational motion

d) Vibratory motion

If an object or particle executes a *to–and–fro* motion about a fixed point, it is said to be in vibratory motion. This is sometimes also called oscillatory motion.

Examples

- Vibration of a string on a guitar
- Movement of a swing

These motions are shown in Figure 2.7

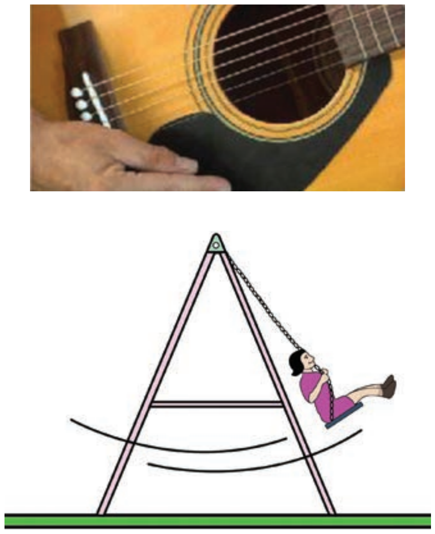


Figure 2.7 **Examples of Vibratory motion**

Other types of motion like elliptical motion and helical motion are also possible.

Motion in One, Two and Three Dimensions

Let the position of a particle in space be expressed in terms of rectangular coordinates x, y and z. When these coordinates change with time, then the particle is said to be in motion. However, it is not necessary that all the three coordinates should together change with time. Even if one or two coordinates change with time, the particle is said to be in motion. Then we have the following classification.

(i) Motion in one dimension

One dimensional motion is the motion of a particle moving along a straight line.

This motion is sometimes known as rectilinear or linear motion.

In this motion, only one of the three rectangular coordinates specifying the position of the object changes with time.

For example, if a car moves from position A to position B along *x*–direction, as shown in Figure 2.8, then a variation in x–coordinate alone is noticed.

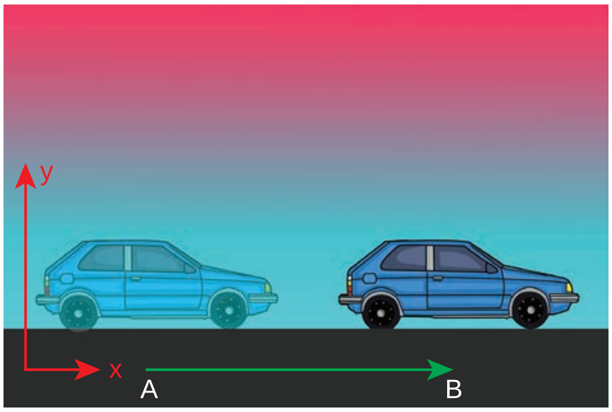


Figure 2.8 **Motion of a particle along one dimension**

Examples

- Motion of a train along a straight railway track.
- An object falling freely under gravity close to Earth.

(ii) Motion in two dimensions

If a particle is moving along a curved path in a plane, then it is said to be in two dimensional motion.

In this motion, two of the three rectangular coordinates specifying the position of object change with time.

For instance, when a particle is moving in the $y - z$ plane, x does not vary, but y and z vary as shown in Figure 2.9

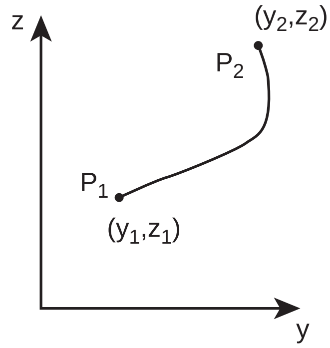


Figure 2.9 **Motion of a particle along two dimensions**

Examples

- Motion of a coin on a carrom board.
- An insect crawling over the floor of a room.

(iii) Motion in three dimensions

A particle moving in usual three dimensional space has three dimensional motion.

In this motion, all the three coordinates specifying the position of an object change with respect to time. When a particle moves in three dimensions, all the three coordinates x, y and z will vary.

Examples

- A bird flying in the sky.
- Random motion of a gas molecule.
- Flying of a kite on a windy day.

2.3 ELEMENTARY CONCEPTS OF VECTOR ALGEBRA

In physics, some quantities possess only magnitude and some quantities possess both magnitude and direction. To understand these physical quantities, it is very important to know the properties of vectors and scalars.

Scalar

It is a property which can be described only by magnitude. In physics a number of quantities can be described by scalars.

Examples

Distance, mass, temperature, speed and energy.

Vector

It is a quantity which is described by both magnitude and direction. Geometrically a vector is a directed line segment which is shown in Figure 2.10. In physics certain quantities can be described only by vectors.

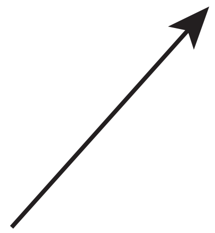


Figure 2.10 Geometrical representation of a vector

Examples

Force, velocity, displacement, position vector, acceleration, linear momentum and angular momentum.

2.3.1 Magnitude of a Vector

The length of a vector is called magnitude of the vector. It is always a positive quantity. Sometimes the magnitude of a vector is also called 'norm' of the vector. For a vector $\vec{A}$, the magnitude or norm is denoted by $|\vec{A}|$ or simply 'A' (Figure 2.11).

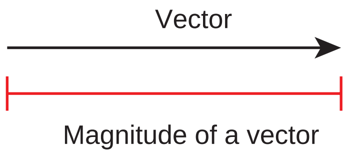


Figure 2.11 Magnitude of a vector

2.3.2 Different types of Vectors

1. **Equal vectors:** Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal when they have equal magnitude and same direction and represent the same physical quantity (Figure 2.12.).

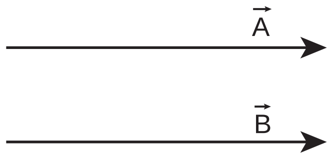


Figure 2.12 Geometrical representation of equal vectors

(a) **Collinear vectors**: Collinear vectors are those which act along the same line. The angle between them can be 0° or 180°.

(i) **Parallel Vectors**: If two vectors $\vec{A}$ and $\vec{B}$ act in the same direction along the same line or on parallel lines, then the angle between them is 0^0 (Figure 2.13).

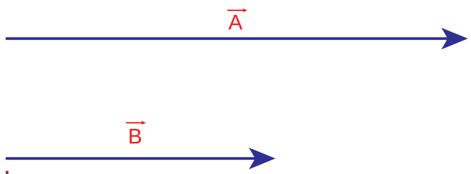


Figure 2.13 **Geometrical representation of parallel vectors**

(ii) Anti–parallel vectors: Two vectors $\vec{A}$ and $\vec{B}$ are said to be anti–parallel when they are in opposite directions along the same line or on parallel lines. Then the angle between them is 180^0 (Figure 2.14).

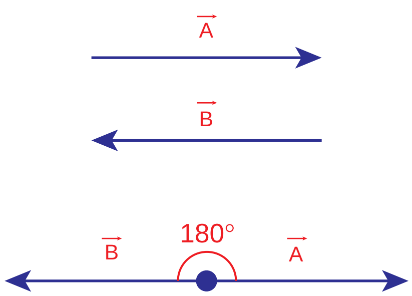


Figure 2.14 **Geometrical representation of anti–parallel vectors.**

2. **Unit vector**: A vector divided by its magnitude is a unit vector. The unit vector for $\vec{A}$ is denoted by $\hat{A}$ (read as A cap or A hat). It has a magnitude equal to unity or one.

$$\text{Since, } \hat{A} = \frac{\vec{A}}{A} \text{ we can write } \vec{A} = A\hat{A}$$

Thus, we can say that the unit vector specifies only the direction of the vector quantity.

3. **Orthogonal unit vectors**: Let $\hat{i}, \hat{j}$ *and* $\hat{k}$ be three unit vectors which specify the directions along positive *x*–axis, positive *y*–axis and positive *z*–axis respectively. These three unit vectors are directed perpendicular to each other, the angle between any two of them is 90°. $\hat{i}, \hat{j}$ and $\hat{k}$ are examples of orthogonal vectors. Two vectors which are perpendicular to each other are called orthogonal vectors as is shown in the Figure 2.15

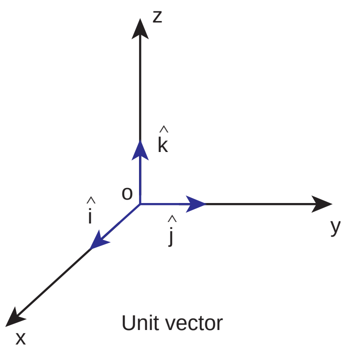


Figure 2.15 **Orthogonal unit vectors**

2.3.3 Addition of Vectors

Since vectors have both magnitude and direction they cannot be added by the method of ordinary algebra. Thus, vectors can be added geometrically or analytically using certain rules called 'vector algebra'. In order to find the sum (resultant) of two vectors, which are inclined to each other, we use (i) Triangular law of addition method or (ii) Parallelogram law of vectors.

Triangular Law of addition method

Let us consider two vectors $\vec{A}$ and $\vec{B}$ as shown in Figure 2.16.

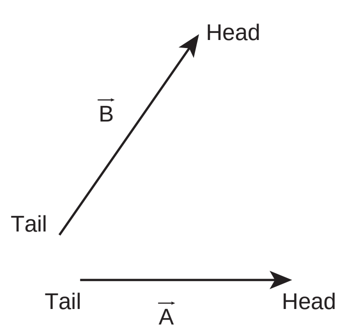


Figure 2.16 **Head and tail of vectors**

To find the resultant of the two vectors we apply the triangular law of addition as follows:

Represent the vectors $\vec{A}$ and $\vec{B}$ by the two adjacent sides of a triangle taken in the same order. Then the resultant is given by the third side of the triangle taken in the reverse order as shown in Figure 2.17.

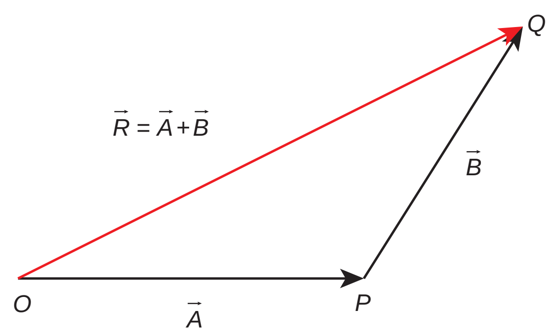


Figure 2.17 **Triangle law of addition**

To explain further, the head of the first vector $\vec{A}$ is connected to the tail of the second vector $\vec{B}$. Let θ be the angle between $\vec{A}$ and $\vec{B}$. Then $\vec{R}$ is the resultant vector connecting the tail of the first vector $\vec{A}$ to the head of the second vector $\vec{B}$. The magnitude of $\vec{R}$ (resultant) is given geometrically by the length of $\vec{R}$ (OQ) and the direction of the resultant vector is the angle between $\vec{R}$ and $\vec{A}$. Thus we write $\vec{R}=\vec{A}+\vec{B}$.

$$\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$$

(1) Magnitude of resultant vector

The magnitude and angle of the resultant vector are determined as follows.

From Figure 2.18, consider the triangle ABN, which is obtained by extending the side OA to ON. ABN is a right angled triangle.

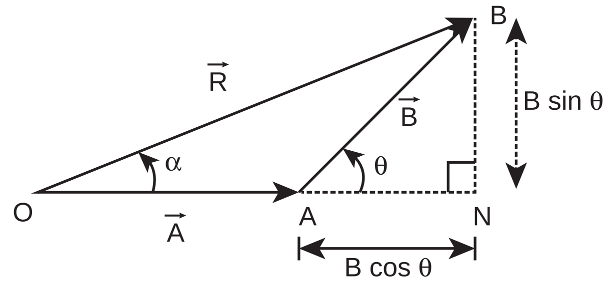


Figure 2.18 **Resultant vector and its direction by triangle law of addition.**

From Figure 2.18

$$\cos\theta=\frac{AN}{B}\therefore AN=B\cos\theta \text{ and}$$

$$\sin\theta=\frac{BN}{B}\quad\therefore BN=B\sin\theta$$

For ΔOBN, we have $OB^2=ON^2+BN^2$

$$\Rightarrow R^2=(A+B\cos\theta)^2+(B\ \sin\theta)^2$$

$$\Rightarrow R^2=A^2+B^2\cos^2\theta+2AB\cos\theta+B^2\sin^2\theta$$

$$\Rightarrow R^2=A^2+B^2\left(\cos^2\theta+\sin^2\theta\right)+2AB\cos\theta$$

$$\Rightarrow R=\sqrt{A^2+B^2+2AB\cos\theta}$$

which is the magnitude of the resultant of $\vec{A}$ and $\vec{B}$

(2) Direction of resultant vectors: If θ is the angle between $\vec{A}$ and $\vec{B}$, then

$$\left|\vec{A}+\vec{B}\right|=\sqrt{A^2+B^2+2AB\cos\theta}\quad(2.1)$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN,

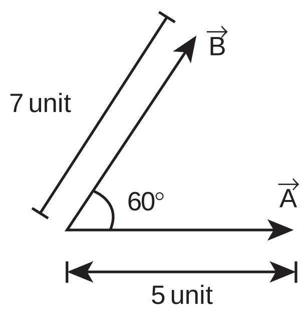
$$\tan\alpha=\frac{BN}{ON}=\frac{BN}{OA+AN}$$

$$\tan\alpha=\frac{B\sin\theta}{A+B\cos\theta}$$

$$\Rightarrow\alpha=\tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)$$

EXAMPLE 2.1

Two vectors $\vec{A}$ and $\vec{B}$ of magnitude 5 units and 7 units respectively make an angle 60° with each other as shown below. Find the magnitude of the resultant vector and its direction with respect to the vector $\vec{A}$.

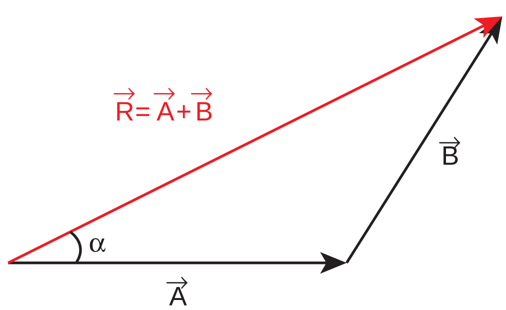


Solution

By following the law of triangular addition, the resultant vector is given by

$$\vec{R} = \vec{A} + \vec{B}$$

as illustrated below



The magnitude of the resultant vector $\vec{R}$ is given by

$$R = \left|\vec{R}\right| = \sqrt{5^2 + 7^2 + 2\times 5\times 7\cos 60^\circ}$$

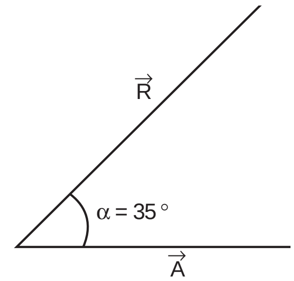
$$R = \sqrt{25 + 49 + \frac{70\times 1}{2}} = \sqrt{109}\ units$$

The angle α between $\vec{R}$ and $\vec{A}$ is given by

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} \qquad (2.2)$$

$$\tan\alpha = \frac{7\times\sin 60^\circ}{5 + 7\cos 60^\circ} = \frac{7\sqrt{3}}{10+7} = \frac{7\sqrt{3}}{17} \cong 0.713$$

$$\therefore \alpha \cong 35^\circ$$



Note Another method to determine the resultant and angle of resultant of two vectors is the Parallelogram Law of vector addition method. It is given in appendix 2.1

2.3.4 Subtraction of vectors

Since vectors have both magnitude and direction two vectors cannot be subtracted from each other by the method of ordinary algebra. Thus, this subtraction can be done either geometrically or analytically. We shall now discuss subtraction of two vectors geometrically using the Figure 2.19

For two non-zero vectors $\vec{A}$ and $\vec{B}$ which are inclined to each other at an angle θ, the difference $\vec{A} - \vec{B}$ is obtained as follows. First obtain $-\vec{B}$ as in Figure 2.19. The angle between $\vec{A}$ and $-\vec{B}$ is 180–θ.

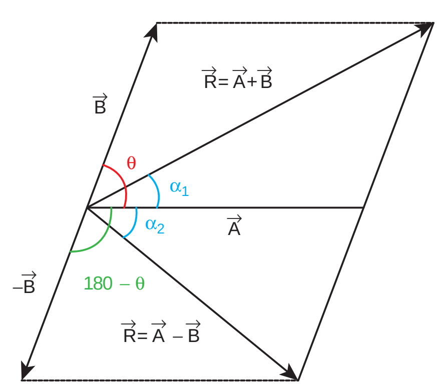


Figure 2.19 Subtraction of vectors

The difference $\vec{A}-\vec{B}$ is the same as the resultant of $\vec{A}$ and $-\vec{B}$.

We can write $\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$ and using the equation (2.1), we have

$$\left|\vec{A}-\vec{B}\right|=\sqrt{A^2+B^2+2AB\cos(180-\theta)} \quad (2.3)$$

Since, $\cos\left(180-\theta\right)=-\cos\theta$, we get

$$\Rightarrow\left|\vec{A}-\vec{B}\right|=\sqrt{A^2+B^2-2AB\cos\theta} \quad (2.4)$$

Again from the Figure 2.19, and using an equation similar to equation (2.2) we have

$$\tan\alpha_2=\frac{B\sin\left(180^\circ-\theta\right)}{A+B\cos\left(180^\circ-\theta\right)} \quad (2.5)$$

But $\sin\left(180^\circ-\theta\right)=\sin\theta$ hence we get

$$\Rightarrow\tan\alpha_2=\frac{B\sin\theta}{A-B\cos\theta} \quad (2.6)$$

Thus the difference $\vec{A}-\vec{B}$ is a vector with magnitude and direction given by equations 2.4 and 2.6 respectively.

EXAMPLE 2.2

Two vectors $\vec{A}$ and $\vec{B}$ of magnitude 5 units and 7 units make an angle 60° with each other. Find the magnitude of the difference vector $\vec{A}-\vec{B}$ and its direction with respect to the vector $\vec{A}$.

Solution

Using the equation (2.4),

$$\left|\vec{A}-\vec{B}\right|=\sqrt{5^2+7^2-2\times5\times7\cos60^\circ}$$

$$=\sqrt{25+49-35}=\sqrt{39}\ units$$

The angle that $\vec{A}-\vec{B}$ makes with the vector $\vec{A}$ is given by

$$\tan\alpha_2=\frac{7\sin60^\circ}{5-7\cos60^\circ}=\frac{7\sqrt{3}}{10-7}=\frac{7}{\sqrt{3}}=4.041$$

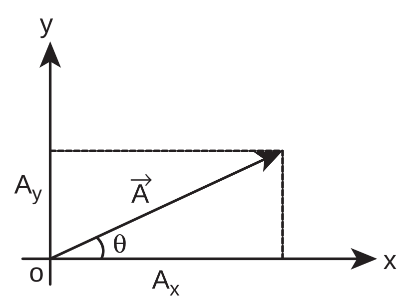
$$\alpha_2=\tan^{-1}\left(4.041\right)\cong76^\circ$$

2.4 COMPONENTS OF A VECTOR

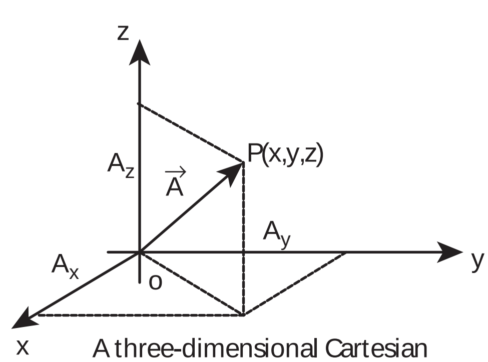
In the Cartesian coordinate system any vector $\vec{A}$ can be resolved into three components along *x*, *y* and *z* directions. This is shown in Figure 2.20.

Consider a 3–dimensional coordinate system. With respect to this a vector can be written in component form as

$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$$



A two-dimensional Cartesian coordinate system



A three-dimensional Cartesian coordinate system

Figure 2.20 Components of a vector in 2 dimensions and 3 dimensions

Here A_x is the x–component of $\vec{A}$, A_y is the y–component of $\vec{A}$ and A_z is the z component of $\vec{A}$.

In a 2–dimensional Cartesian coordinate system (which is shown in the Figure 2.20) the vector $\vec{A}$ is given by

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

If $\vec{A}$ makes an angle θ with x axis, and A_x and A_y are the components of $\vec{A}$ along x–axis and y–axis respectively, then as shown in Figure 2.21,

$$A_x = A\cos\theta,\ A_y = A\sin\theta$$

where 'A' is the magnitude (length) of the vector $\vec{A}$, $A = \sqrt{A_x^2 + A_y^2}$

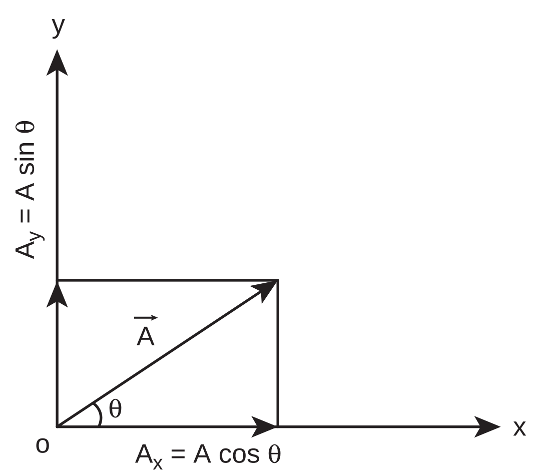


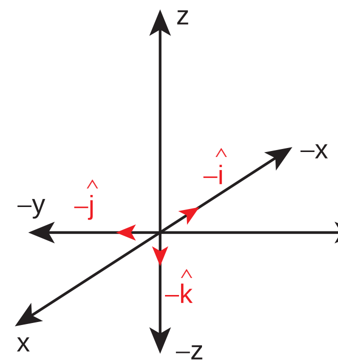
Figure 2.21 Resolution of a vector

EXAMPLE 2.3

What are the unit vectors along the negative x–direction, negative y–direction, and negative z– direction?

Solution

The unit vectors along the negative directions can be shown as in the following figure.



Then we have:

The unit vector along the negative x direction = $-\hat{i}$

The unit vector along the negative y direction = $-\hat{j}$

The unit vector along the negative z direction =$-\hat{k}$.

2.4.1 Vector addition using components

In the previous section we have learnt about addition and subtraction of two vectors using geometric methods. But once we choose a coordinate system, the addition and subtraction of vectors becomes much easier to perform.

The two vectors $\vec{A}$ and $\vec{B}$ in a Cartesian coordinate system can be expressed as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Then the addition of two vectors is equivalent to adding their corresponding x, y and z components.

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Similarly the subtraction of two vectors is equivalent to subtracting the corresponding x, y and z components.

$$\vec{A}-\vec{B}=\left(A_x-B_x\right)\hat{i}+(A_y-B_y)\hat{j}+\left(A_z-B_z\right)\hat{k}$$

The above rules form an analytical way of adding and subtracting two vectors.

EXAMPLE 2.4

Two vectors $\vec{A}$ and $\vec{B}$ are given in the component form as $\vec{A}=5\hat{i}+7\hat{j}-4\hat{k}$ and $\vec{B}=6\hat{i}+3\hat{j}+2\hat{k}$. Find $\vec{A}+\vec{B}$, $\vec{B}+\vec{A}$, $\vec{A}-\vec{B}$, $\vec{B}-\vec{A}$

Solution

$$\vec{A}+\vec{B}=(5\hat{i}+7\hat{j}-4\hat{k})+(6\hat{i}+3\hat{j}+2\hat{k})$$
$$=11\hat{i}+10\hat{j}-2\hat{k}$$
$$\vec{B}+\vec{A}=(6\hat{i}+3\hat{j}+2\hat{k})+(5\hat{i}+7\hat{j}-4\hat{k})$$
$$=(6+5)\hat{i}+(3+7)\hat{j}+(2-4)\hat{k}$$
$$=11\hat{i}+10\hat{j}-2\hat{k}$$
$$\vec{A}-\vec{B}=(5\hat{i}+7\hat{j}-4\hat{k})-(6\hat{i}+3\hat{j}+2\hat{k})$$
$$=-\hat{i}+4\hat{j}-6\hat{k}$$
$$\vec{B}-\vec{A}=\hat{i}-4\hat{j}+6\hat{k}$$

Note that the vectors $\vec{A}+\vec{B}$ and $\vec{B}+\vec{A}$ are same and the vectors $\vec{A}-\vec{B}$ and $\vec{B}-\vec{A}$ are opposite to each other.

Note The addition of two vectors using components depends on the choice of the coordinate system. But the geometric way of adding and subtracting two vectors is independent of the coordinate system used.

2.5 MULTIPLICATION OF VECTOR BY A SCALAR

A vector $\vec{A}$ multiplied by a scalar λ results in another vector, $\lambda\vec{A}$. If λ is a positive number then $\lambda\vec{A}$ is also in the direction of $\vec{A}$. If λ is a negative number, $\lambda\vec{A}$ is in the opposite direction to the vector $\vec{A}$.

EXAMPLE 2.5

Given the vector $\vec{A}=2\hat{i}+3\hat{j}$, what is 3 $\vec{A}$?

Solution

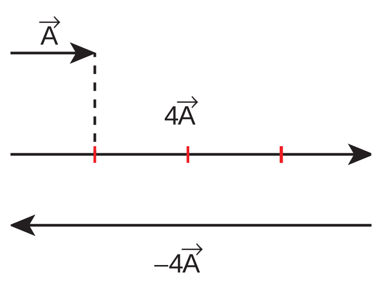
$$3\vec{A}=3\left(2\hat{i}+3\hat{j}\right)=6\hat{i}+9\hat{j}$$

The vector $3\vec{A}$ is in the same direction as vector $\vec{A}$.

EXAMPLE 2.6

A vector $\vec{A}$ is given as in the following Figure. Find $4\vec{A}$ and $-4\vec{A}$

Solution



In physics, certain vector quantities can be defined as a scalar times another vector quantity.

For example

1) Force $\vec{F}=m\vec{a}$. Here mass 'm' is a scalar, and $\vec{a}$ is the acceleration. Since 'm' is always a positive scalar, the direction

of force is always in the direction of acceleration.

2) Linear momentum $\vec{P}$ = m$\vec{v}$. Here $\vec{v}$ is the velocity. The direction of linear momentum is also in the direction of velocity.

3) Force $\vec{F} = q\vec{E}$, Here the electric charge 'q' is a scalar, and $\vec{E}$ is the electric field. Since charge can be positive or negative, the direction of force $\vec{F}$ is correspondingly either in the direction of $\vec{E}$ or opposite to the direction of $\vec{E}$.

2.5.1 Scalar Product of Two Vectors

Definition

The scalar product (or dot product) of two vectors is defined as the product of the magnitudes of both the vectors and the cosine of the angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having an angle θ between them, then their scalar product is defined as $\vec{A}\cdot\vec{B} = AB\cos\theta$. Here, A and B are magnitudes of $\vec{A}$ and $\vec{B}$.

Properties

(i) The product quantity $\vec{A}\cdot\vec{B}$ is always a scalar. It is positive if the angle between the vectors is acute (i.e., θ < 90°) and negative if the angle between them is obtuse (i.e. 90°<θ< 180°).

(ii) The scalar product is commutative, i.e. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(iii) The vectors obey distributive law i.e. $\vec{A}.\left(\vec{B}+\vec{C}\right) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

(iv) The angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$

(v) The scalar product of two vectors will be maximum when cos θ = 1, i.e. θ = 0°, i.e., when the vectors are parallel;

$$(\vec{A}\cdot\vec{B})_{max} = AB$$

(vi) The scalar product of two vectors will be minimum, when cos θ = −1, i.e. θ = 180°

$(\vec{A}\cdot\vec{B})_{min}$ = −AB, when the vectors are anti-parallel.

(vii) If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other then their scalar product $\vec{A}\cdot\vec{B} = 0$, because cos 90°=0. Then the vectors $\vec{A}$ and $\vec{B}$ are said to be mutually orthogonal.

(viii) The scalar product of a vector with itself is termed as self–dot product and is given by $\left(\vec{A}\right)^2 = \vec{A}\cdot\vec{A} = AA\cos\theta = A^2$. Here angle $\theta = 0°$

The magnitude or norm of the vector $\vec{A}$ is $\left|\vec{A}\right| = A = \sqrt{\vec{A}\cdot\vec{A}}$

(ix) In case of a unit vector $\hat{n}$

$\hat{n}\cdot\hat{n}$ = 1 x 1 x cos 0 = 1. For example, $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(x) In the case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$,

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 1\cdot 1\cos 90° = 0$$

(xi) In terms of components the scalar product of $\vec{A}$ and $\vec{B}$ can be written as

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

$= A_xB_x + A_yB_y + A_zB_z$, with all other terms zero.

The magnitude of vector $\left|\vec{A}\right|$ is given by

$$\left|\vec{A}\right| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

EXAMPLE 2.7

Given two vectors $\vec{A} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{B} = \hat{i} + 3\hat{j} + 6\hat{k}$, Find the product $\vec{A}.\vec{B}$, and the magnitudes of $\vec{A}$ and $\vec{B}$. What is the angle between them?

Solution

$$\vec{A}.\vec{B} = 2 + 12 + 30 = 44$$

$$\text{Magnitude A} = \sqrt{4+16+25} = \sqrt{45} \text{ units}$$

$$\text{Magnitude B} = \sqrt{1+9+36} = \sqrt{46} \text{ units}$$

The angle between the two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$$

$$= \cos^{-1}\left(\frac{44}{\sqrt{45 \times 46}}\right) = \cos^{-1}\left(\frac{44}{45.49}\right)$$

$$= \cos^{-1}(0.967)$$

$$\therefore \theta \cong 15^\circ$$

EXAMPLE 2.8

Check whether the following vectors are orthogonal.

i) $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 4\hat{i} - 5\hat{j}$

ii) $\vec{C} = 5\hat{i} + 2\hat{j}$ and $\vec{D} = 2\hat{i} - 5\hat{j}$

Solution

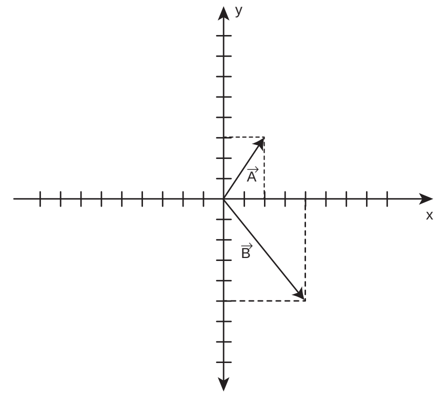
$$\vec{A}.\vec{B} = 8 - 15 = -7 \neq 0$$

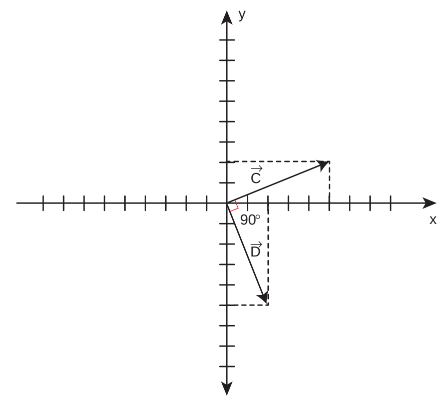
Hence $\vec{A}$ and $\vec{B}$ are not orthogonal to each other.

$$\vec{C}.\vec{D} = 10 - 10 = 0$$

Hence, $\vec{C}$ and $\vec{D}$ are orthogonal to each other.

It is also possible to geometrically show that the vectors $\vec{C}$ *and* $\vec{D}$ are orthogonal to each other. This is shown in the following Figure.





In physics, the work done by a force $\vec{F}$ to move an object through a small displacement $d\vec{r}$ is defined as,

$$W = \vec{F}.d\vec{r}$$

$$W = F\ dr\ \cos\theta$$

The work done is basically a scalar product between the force vector and the displacement vector. Apart from work done, there are other physical quantities which are also defined through scalar products.

Note In uniform circular motion, the centripetal force is perpendicular to the displacement. Hence, work done by this force is zero.

2.5.2 The Vector Product of Two Vectors

Definition

The vector product or cross product of two vectors is defined as another vector having a magnitude equal to the product of the magnitudes of two vectors and the sine of the angle between them. The direction of the product vector is perpendicular to the plane containing the two vectors, in accordance with the right hand screw rule or right hand thumb rule (Figure 2.22).

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product is written as $\vec{A}\times\vec{B}$ which is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A}\times\vec{B} = (AB\sin\theta)\hat{n}$$

The direction $\hat{n}$ of $\vec{A}\times\vec{B}$, i.e., $\vec{C}$ is perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ and is in the sense of advancement of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right-handed screw whose axis is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$, is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A}\times\vec{B}$ i.e. $\vec{C}$ which is illustrated in Figure 2.22.

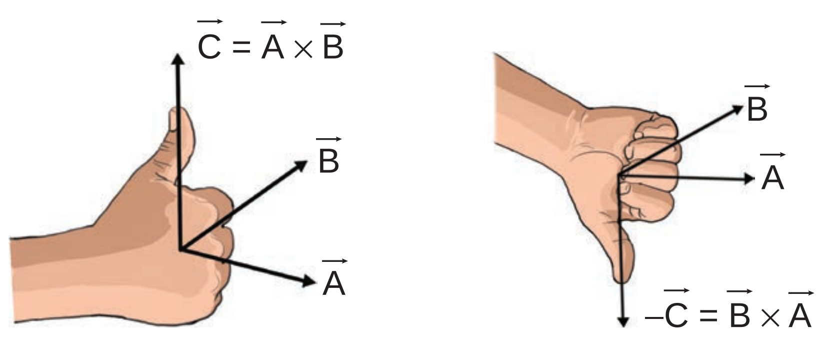
Note

According to Right Hand Rule, if the curvature of the fingers of the right hand represents the sense of rotation of the object, then the thumb, held perpendicular to the curvature of the fingers, represents the direction of the resultant $\vec{C}$.

VECTOR PRODUCT ("CROSS" PRODUCT)

The vector product of $\vec{A}$ and $\vec{B}$, written as $\vec{A}\times\vec{B}$, produces a third vector $\vec{C}$

$$\vec{C} = \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n} \qquad -\vec{C} = \vec{B}\times\vec{A}$$



$$\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$$

Figure 2.22 **Vector product of two vectors**

Properties of vector (cross) product.

(i) The vector product of any two vectors is always another vector whose direction is perpendicular to the plane containing these two vectors, i.e., orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, even though the vectors $\vec{A}$ and $\vec{B}$ may or may not be mutually orthogonal.

(ii) The vector product of two vectors is not commutative, i.e., $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ But., $\vec{A}\times\vec{B} = -[\vec{B}\times\vec{A}]$

Here it is worthwhile to note that $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB \sin\theta$ i.e., in the case of the product vectors $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$, the magnitudes are equal but directions are opposite to each other.

(iii) The vector product of two vectors will have maximum magnitude when $\sin\theta = 1$, i.e., $\theta = 90°$ i.e., when the vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other.

$$\left(\vec{A}\times\vec{B}\right)_{max} = AB\hat{n}$$

(iv) The vector product of two non–zero vectors will be minimum when $|\sin\theta| = 0$, $i.e., \theta = 0°$ or $180°$

$$\left(\vec{A}\times\vec{B}\right)_{min} = 0$$

i.e., the vector product of two non–zero vectors vanishes, if the vectors are either parallel or antiparallel.

(v) The self–cross product, i.e., product of a vector with itself is the null vector

$$\vec{A}\times\vec{A} = AA\sin 0°\hat{n} = \vec{0}.$$

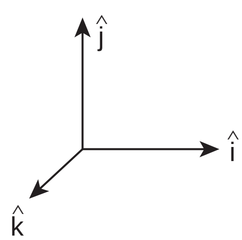
In physics the null vector $\vec{0}$ is simply denoted as zero.

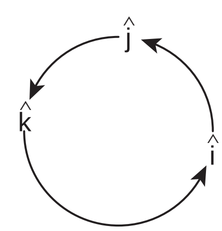
(vi) The self–vector products of unit vectors are thus zero.

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$$

(vii) In the case of orthogonal unit vectors, $\hat{i}, \hat{j}, \hat{k}$, in accordance with the right hand screw rule:

$$\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i} \text{ and } \hat{k}\times\hat{i} = \hat{j}$$





Also, since the cross product is not commutative,

$$\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}$$
$$\text{and } \hat{i}\times\hat{k} = -\hat{j}$$

(viii) In terms of components, the vector product of two vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\left(A_y B_z - A_z B_y\right)$$
$$+\hat{j}\left(A_z B_x - A_x B_z\right)$$
$$+\hat{k}\left(A_x B_y - A_y B_x\right)$$

Note that in the $\hat{j}^{th}$ component the order of multiplication is different than $\hat{i}^{th}$ and $\hat{k}^{th}$ components.

(ix) If two vectors $\vec{A}$ *and* $\vec{B}$ form adjacent sides in a parallelogram, then the magnitude of $\vec{A}\times\vec{B}$ will give the area of the parallelogram as represented graphically in Figure 2.23.

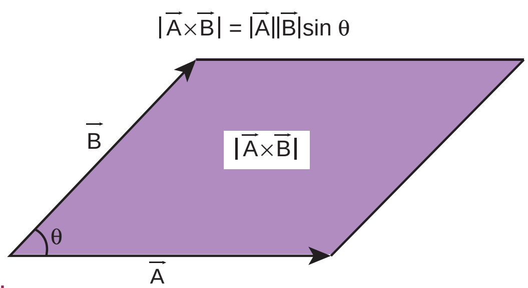


Figure 2.23 Area of parallelogram

(x) Since we can divide a parallelogram into two equal triangles as shown in the Figure 2.24, the area of a triangle with $\vec{A}$ *and* $\vec{B}$ as sides is $\frac{1}{2}\left|\vec{A}\times\vec{B}\right|$. This is shown in the Figure 2.24. (This fact will be used when we study Kepler's laws in unit 6)

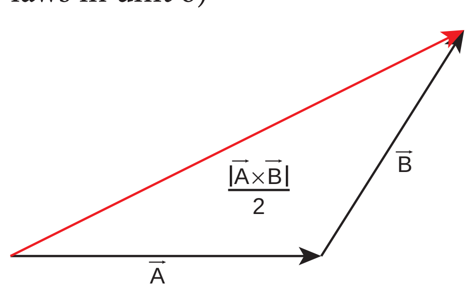


Figure 2.24 Area of triangle

A number of quantities used in Physics are defined through vector products. Particularly physical quantities representing rotational effects like torque, angular momentum, are defined through vector products.

Examples

(i) Torque $\vec{\tau} = \vec{r}\times\vec{F}$. where $\vec{F}$ is Force and $\vec{r}$ is position vector of a particle

(ii) Angular momentum $\vec{L} = \vec{r}\times\vec{p}$ where $\vec{p}$ is the linear momentum

(iii) Linear Velocity $\vec{v} = \vec{\omega}\times\vec{r}$ where $\vec{\omega}$ is angular velocity

EXAMPLE 2.9

Two vectors are given as $\vec{r} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{F} = 3\hat{i} - 2\hat{j} + 4\hat{k}$. Find the resultant vector $\vec{\tau} = \vec{r}\times\vec{F}$

Solution

$$\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 3 & -2 & 4 \end{vmatrix}$$

$$\vec{\tau} = (12 - (-10)\hat{i} + (15 - 8)\hat{j} + (-4 - 9)\hat{k}$$

$$\vec{\tau} = 22\hat{i} + 7\hat{j} - 13\hat{k}$$

2.5.3 Properties of the components of vectors

If two vectors $\vec{A}$ and $\vec{B}$ are equal, then their individual components are also equal.

Let $\vec{A} = \vec{B}$

Then $A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

i.e., $A_x = B_x$, $A_y = B_y$, $A_z = B_z$

EXAMPLE 2.10

Compare the components for the following vector equations

a) $\vec{F} = m\vec{a}$ Here m is positive number

b) $\vec{p} = 0$

Solution

Case (a):

$$\vec{F} = m\vec{a}$$

$$F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = ma_x\hat{i} + ma_y\hat{j} + ma_z\hat{k}$$

By comparing the components, we get

$$F_x = ma_x, F_y = ma_y, F_z = ma_z$$

This implies that one vector equation is equivalent to three scalar equations.

Case (b)

$$\vec{p} = 0$$

$$p_x\hat{i} + p_y\hat{j} + p_z\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

By comparing the components, we get

$$p_x = 0, p_y = 0, p_z = 0$$

EXAMPLE 2.11

Determine the value of the T from the given vector equation.

$$5\hat{j} - T\hat{j} = 6\hat{j} + 3T\hat{j}$$

Solution

By comparing the components both sides, we can write

$$5 - 6 = 3T + T$$

$$-1 = 4T$$

$$T = -\frac{1}{4}$$

EXAMPLE 2.12

Compare the components of vector equation $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{F}_4$

Solution

We can resolve all the vectors in x, y and z components with respect to Cartesian coordinate system.

Once we resolve the components we can separately equate the x components on both sides, y components on both sides, and z components on both the sides of the equation, we then get

$$F_{1x} + F_{2x} + F_{3x} = F_{4x}$$

$$F_{1y} + F_{2y} + F_{3y} = F_{4y}$$

$$F_{1z} + F_{2z} + F_{3z} = F_{4z}$$

2.6 POSITION VECTOR

It is a vector which denotes the position of a particle at any instant of time, with respect to some reference frame or coordinate system.

The position vector $\vec{r}$ of the particle at a point P is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where x, y and z are components of $\vec{r}$, Figure 2.25 shows the position vector $\vec{r}$.

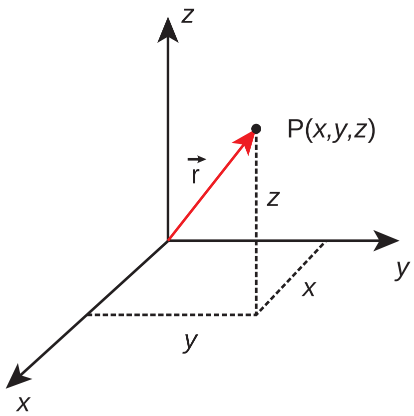
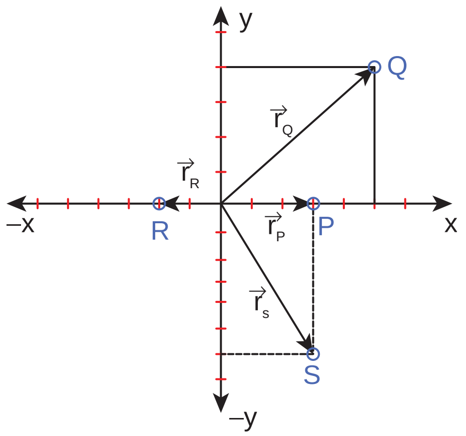


Figure 2.25 Position vector in Cartesian coordinate system

EXAMPLE 2.13

Determine the position vectors for the following particles which are located at points P, Q, R, S.



Solution

The position vector for the point P is

$$\vec{r}_P = 3\hat{i}$$

The position vector for the point Q is

$$\vec{r}_Q = 5\hat{i} + 4\hat{j}$$

The position vector for the point R is

$$\vec{r}_R = -2\hat{i}$$

The position vector for the point S is

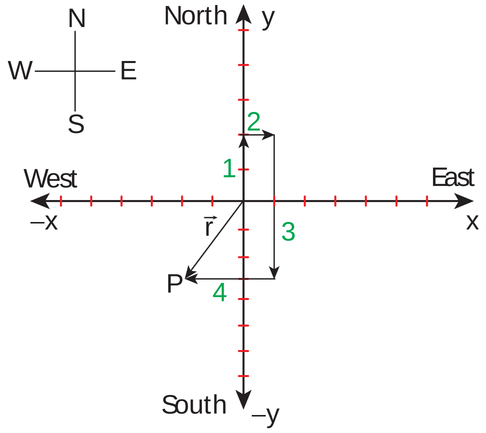
$$\vec{r}_s = 3\hat{i} - 6\hat{j}$$

EXAMPLE 2.14

A person initially at rest starts to walk 2 m towards north, then 1 m towards east, then 5 m towards south and then 3 m towards west. What is the position vector of the person at the end of the trip?

Solution

As shown in the Figure, the positive x axis is taken as east direction, positive y direction is taken as north.



After the trip, the person reaches the point P whose position vector given by

$$\vec{r} = -2\hat{i} - 3\hat{j}$$

The displacement direction is south west.

2.7 DISTANCE AND DISPLACEMENT

Distance is the actual path length travelled by an object in the given interval of time during the motion. It is a positive scalar quantity.

Displacement is the difference between the final and initial positions of the object in a given interval of time. It can also be defined as the shortest distance between these two positions of the object and its direction is from the initial to final position of the object, during the given interval of time. It is a vector quantity. Figure 2.26 illustrates the difference between displacement and distance.

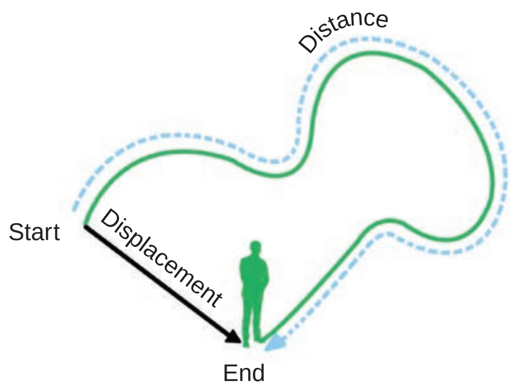
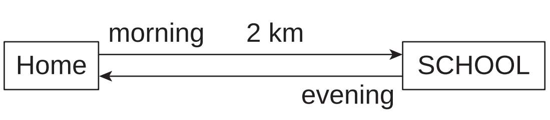


Figure 2.26 Distance and displacement

EXAMPLE 2.15

Assume your school is located 2 km away from your home. In the morning you are going to school and in the evening you come back home. In this entire trip what is the distance travelled and the displacement covered?

Solution



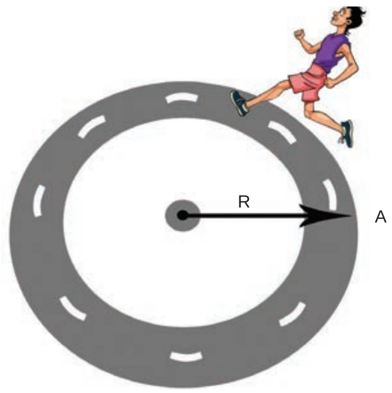
The displacement covered is zero. It is because your initial and final positions are the same.

But the distance travelled is 4 km.

EXAMPLE 2.16

An athlete covers 3 rounds on a circular track of radius 50 m. Calculate the total distance and displacement travelled by him.

Solution



The total distance the athlete covered =3x circumference of track

$$\text{Distance} = 3 \times 2\pi \times 50\ m$$
$$= 300\pi\ \ m \quad \text{(or)}$$
$$\text{Distance} \approx 300 \text{ x } 3.14 \approx 942\ m$$

The displacement is zero, since the athlete reaches the same point A after three rounds from where he started.

2.7.1 Displacement Vector in Cartesian Coordinate System

In terms of position vector, the displacement vector is given as follows. Let us consider a particle moving from a point P_1 having position vector $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ to a point P_2 where its position vector is $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

The displacement vector is given by

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

This displacement is also shown in Figure 2.27.

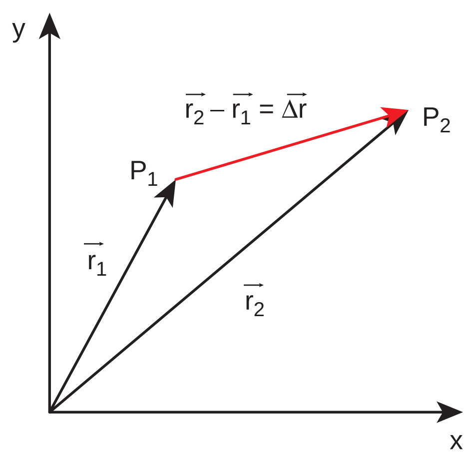
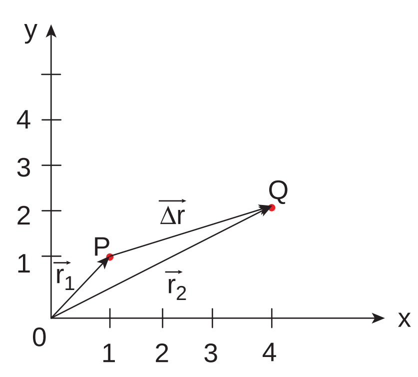


Figure 2.27 Displacement vector

EXAMPLE 2.17

Calculate the displacement vector for a particle moving from a point P to Q as shown below. Calculate the magnitude of displacement.



Solution

The displacement vector $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, with

$$\vec{r}_1 = \hat{i} + \hat{j} \text{ and } \vec{r}_2 = 4\hat{i} + 2\hat{j}$$
$$\therefore \Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \left(4\hat{i} + 2\hat{j}\right) - \left(\hat{i} + \hat{j}\right)$$
$$= \left(4 - 1\right)\hat{i} + \left(2 - 1\right)\hat{j}$$
$$\therefore \Delta\vec{r} = 3\hat{i} + \hat{j}$$

The magnitude of the displacement vector $\Delta r = \sqrt{3^2 + 1^2} = \sqrt{10}$ unit.

Note

(1) The Distance travelled by an object in motion in a given time is never negative or zero, it is always positive.

(2) The displacement of an object, in a given time can be positive, zero or negative.

(3) The displacement of an object can be equal or less than the distance travelled but never greater than distance travelled.

(4) The distance covered by an object between two positions can have many values, but the displacement between them has only one value (in magnitude).

2.8 DIFFERENTIAL CALCULUS

The Concept of a function

1) Any physical quantity is represented by a "function" in mathematics. Take the example of temperature T. We know that the temperature of the surroundings is changing throughout the day. It increases till noon and decreases in the evening. At any

time "t" the temperature T has a unique value. Mathematically this variation can be represented by the notation 'T(t)' and it should be called "temperature as a function of time". It implies that if the value of 't' is given, then the function "T(t)" will give the value of the temperature at that time 't'. Similarly, the position of a bus in motion along the x direction can be represented by x(t) and this is called 'x' as a function of time'. Here 'x' denotes the x coordinate.

Example

Consider a function $f(x) = x^2$. Sometimes it is also represented as $y = x^2$. Here y is called the dependent variable and x is called independent variable. It means as x changes, y also changes. Once a physical quantity is represented by a function, one can study the variation of the function over time or over the independent variable on which the quantity depends. Calculus is the branch of mathematics used to analyse the change of any quantity.

If a function is represented by $y = f(x)$, then dy/dx represents the derivative of y with respect to x. Mathematically this represents the variation of y with respect to change in x, for various continuous values of x.

Mathematically the derivative dy/dx is defined as follows

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\frac{dy}{dx}$ represents the limit that the quantity $\frac{\Delta y}{\Delta x}$ attains, as Δx tends to zero.

Graphically this is represented as shown in Figure 2.28.

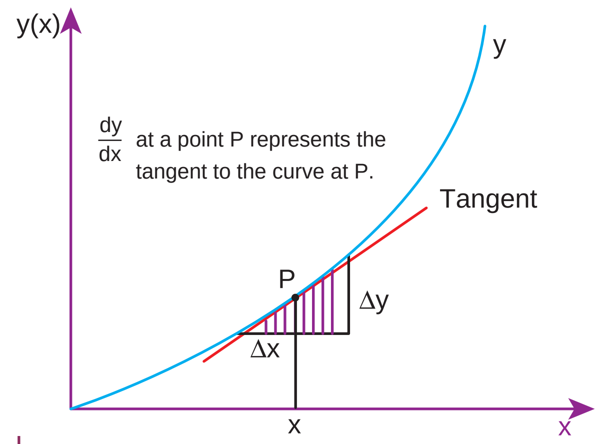


Figure 2.28 Derivative of a function

EXAMPLE 2.18

Consider the function $y = x^2$. Calculate the derivative $\frac{dy}{dx}$ using the concept of limit, at the point $x = 2$.

Solution

Let us take two points given by

$x_1 = 2$ and $x_2 = 3$, then $y_1 = 4$ and $y_2 = 9$

Here $\Delta x = 1$ and $\Delta y = 5$

Then

$$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$$

If we take $x_1 = 2$ and $x_2 = 2.5$, then $y_1 = 4$ and $y_2 = (2.5)^2 = 6.25$

Here $\Delta x = 0.5 = \frac{1}{2}$ and $\Delta y = 2.25$

Then

$$\frac{\Delta y}{\Delta x} = \frac{6.25-4}{0.5} = 4.5$$

If we take $x_1 = 2$ and $x_2 = 2.25$, then $y_1 = 4$ and $y_2 = 5.0625$

$$\text{Here } \Delta x = 0.25 = \frac{1}{4},\ \Delta y = 1.0625$$

$$\frac{\Delta y}{\Delta x} = \frac{5.0625-4}{0.25} = \frac{(5.0625-4)}{\frac{1}{4}}$$
$$= 4(5.0625-4) = 4.25$$

If we take $x_1 = 2$ and $x_2 = 2.1$, then $y_1 = 4$ and $y_2 = 4.41$

$$\text{Here } \Delta x = 0.1 = \frac{1}{10} \text{ and}$$

$$\frac{\Delta y}{\Delta x} = \frac{(4.41-4)}{\frac{1}{10}} = 10(4.41-4) = 4.1$$

These results are tabulated as shown below:

x_1	x_2	Δx	y_1	y_2	$\frac{\Delta y}{\Delta x}$
2	2.25	0.25	4	5.0625	4.25
2	2.1	0.1	4	4.41	4.1
2	2.01	0.01	4	4.0401	4.01
2	2.001	0.001	4	4.004001	4.001
2	2.0001	0.0001	4	4.00040001	4.0001

From the above table, the following inferences can be made.

- As Δx tends to zero, $\frac{\Delta y}{\Delta x}$ approaches the limit given by the number 4.
- At a point x = 2, the derivative $\frac{dy}{dx} = 4$.
- It should also be mentioned here that $\Delta x \to 0$ does not mean that $\Delta x = 0$.

This is because, if we substitute $\Delta x = 0$, $\frac{\Delta y}{\Delta x}$ becomes indeterminate.

In general, we can obtain the derivative of the function $y = x^2$, as follows:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$
$$= \frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

The table below shows the derivatives of some common functions used in physics

Function	Derivative
$y = x$	$dy/dx = 1$
$y = x^2$	$dy/dx = 2x$
$y = x^3$	$dy/dx = 3x^2$
$y = x^n$	$dy/dx = nx^{n-1}$
$y = \sin x$	$dy/dx = \cos x$
$y = \cos x$	$dy/dx = -\sin x$
y = constant	$dy/dx = 0$
$y = AB$	$\frac{dy}{dx} = A\left(\frac{dB}{dx}\right) + \left(\frac{dA}{dx}\right)B$

In physics, velocity, speed and acceleration are all derivatives with respect to time 't'. This will be dealt with in the next section.

EXAMPLE 2.19

Find the derivative with respect to t, of the function $x = A_0 + A_1 t + A_2 t^2$ where A_0, A_1 and A_2 are constants.

Solution

Note that here the independent variable is 't' and the dependent variable is 'x'

The required derivative is $dx/dt = 0+A_1+2A_2t$

The second derivative is $d^2x/dt^2 = 2A_2$

2.9 INTEGRAL CALCULUS

Integration is an area finding process. For certain geometric shapes we can directly find the area. But for irregular shapes the process of integration is used. Consider for example the areas of a rectangle and an irregularly shaped curve, as shown in Figure 2.29.

The area of the rectangle is simply given by A = length × breadth = (b–a) c

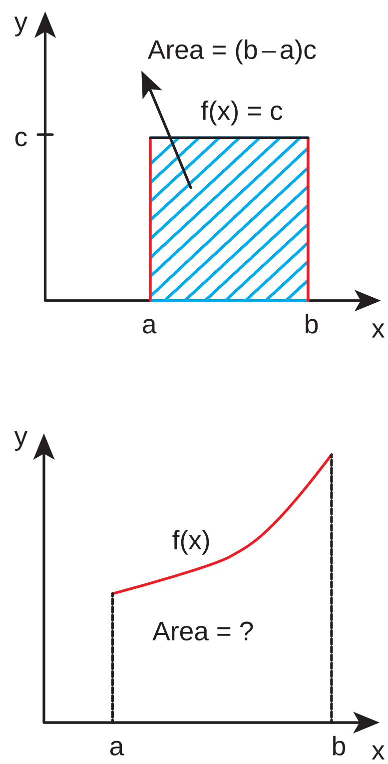


Figure 2.29 Area of rectangular and irregular shape

But to find the area of the irregular shaped curve given by f(x), we divide the area into rectangular strips as shown in the Figure 2.30.

The area under the curve is approximately equal to sum of areas of each rectangular strip.

This is given by $A \approx f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$.

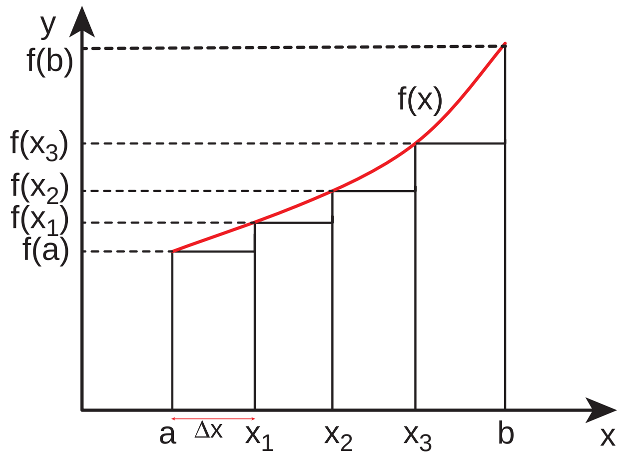


Figure 2.30 Area under the curve using rectangular strip

Where $f(a)$ is the value of the function $f(x)$ at x = a, $f(x_1)$ is the value of $f(x)$ for $x = x_1$ and so on.

As we increase the number of strips, the area evaluated becomes more accurate. If the area under the curve is divided into N strips, the area under the curve is given by

$$A = \sum_{n=1}^{N} f(x_n)\Delta x$$

As the number of strips goes to infinity, $N \to \infty$, the sum becomes an integral,

$$A = \int_a^b f(x)\,dx$$

(Note: As $N \to \infty$, $\Delta x \to 0$)

The integration will give the total area under the curve f (x). This is shown in Figure 2.31.

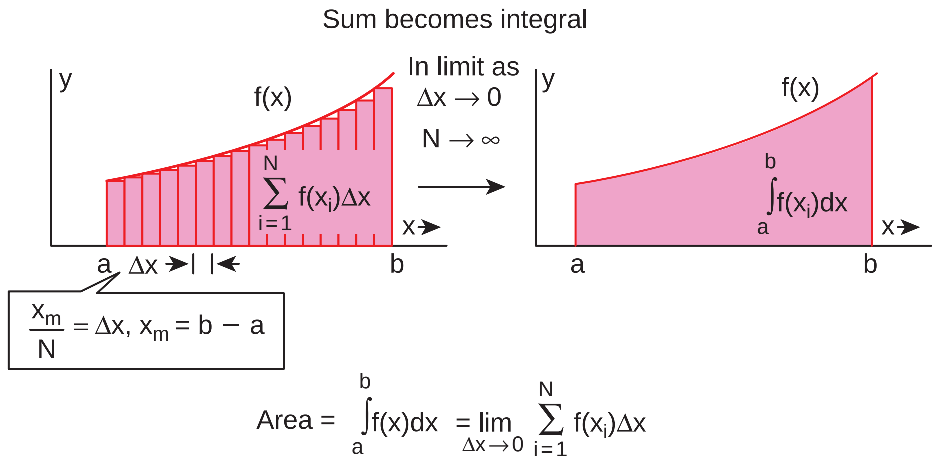


Figure 2.31 Relation between summation and integration

Examples

In physics the work done by a force F(x) on an object to move it from point a to point b in one dimension is given by

$$W = \int_a^b F(x)\,dx$$

(No scalar products is required here, since motion here is in one dimension)

1) The work done is the area under the force displacement graph as shown in Figure 2.32

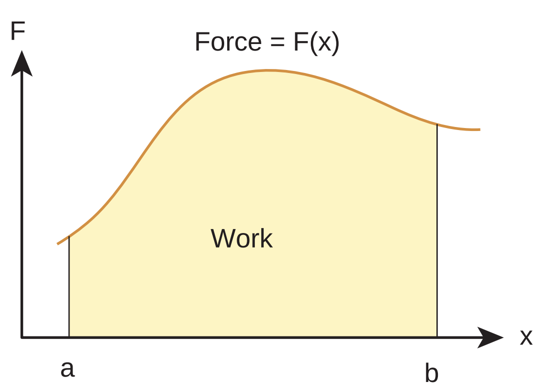


Figure 2.32 Work done by the force

2) The impulse given by the force in an interval of time is calculated between the interval from time t = 0 to time t = t_1 as

$$\text{Impulse } I = \int_0^{t_1} F\,dt$$

The impulse is the area under the force function F (t) - t graph as shown in Figure 2.33.

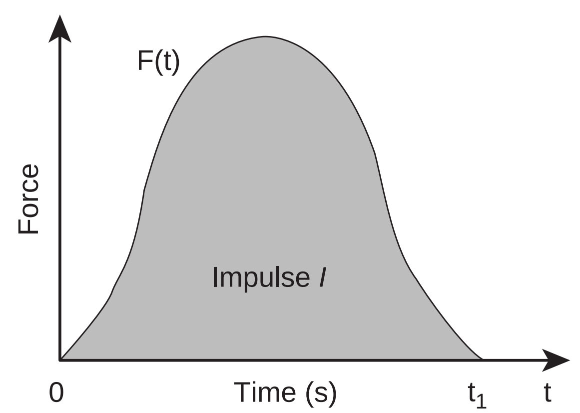


Figure 2.33 Impulse of a force

Average velocity

Consider a particle located initially at point P having position vector $\vec{r}_1$. In a time interval Δt the particle is moved to the point Q having position vector $\vec{r}_2$. The displacement vector is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$. This is shown in Figure 2.34.

The average velocity is defined as ratio of the displacement vector to the corresponding time interval

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}.$$

It is a vector quantity. The direction of average velocity is in the direction of the displacement vector ($\Delta\vec{r}$).

This is also shown in Figure 2.34.

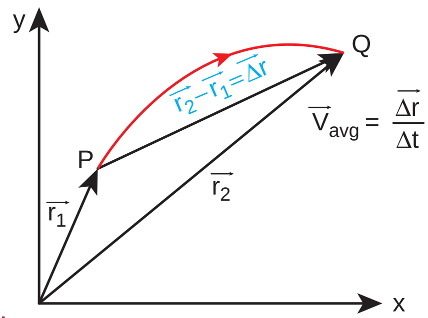


Figure 2.34 Average velocity

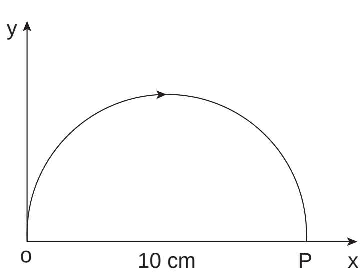
Average speed

The average speed is defined as the ratio of total path length travelled by the particle in a time interval.

Average speed = total path length / total time

EXAMPLE 2.20

Consider an object travelling in a semi-circular path from point O to point P in 5 second, as shown in the Figure given below. Calculate the average velocity and average speed.



Solution

$$\text{Average velocity } \vec{v}_{avg} = \frac{\vec{r}_P - \vec{r}_O}{\Delta t}$$

$$\text{Here } \Delta t = 5\ s$$

$$\vec{r}_O = 0,\ \vec{r}_P = 10\hat{i}$$

$$\vec{v}_{avg} = \frac{10\hat{i}\,cm}{5\,s} = 2\hat{i}\,cm\,s^{-1}.$$

The average velocity is in the positive x direction.

The average speed = total path length / time taken (the path is semi-circular)

$$= \frac{5\pi\,cm}{5\,s} = \pi\,cm\,s^{-1} \approx 3.14\ cm\,s^{-1}$$

Note that the average speed is greater than the magnitude of the average velocity.

Instantaneous velocity or velocity

The instantaneous velocity at an instant t or simply 'velocity' at an instant t is defined as limiting value of the average velocity as $\Delta t \to 0$, evaluated at time t.

In other words, velocity is equal to rate of change of position vector with respect to time. Velocity is a vector quantity.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In component form, this velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}.$$

Here $\frac{dx}{dt} = v_x = x-$ component of velocity

$$\frac{dy}{dt} = v_y = y - \text{component of velocity}$$

$$\frac{dz}{dt} = v_z = z - \text{component of velocity}$$

The magnitude of velocity v is called speed and is given by

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Speed is always a positive scalar. The unit of speed is also meter per second.

EXAMPLE 2.21

The position vector of a particle is given $\vec{r} = 2t\,\hat{i} + 3\,t^2\,\hat{j} - 5\,\hat{k}$.

a) Calculate the velocity and speed of the particle at any instant t

b) Calculate the velocity and speed of the particle at time $t = 2$ s

Solution

The velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = 2\hat{i} + 6t\hat{j}$

The speed $v(t) = \sqrt{2^2 + (6t)^2}\ ms^{-1}$

The velocity of the particle at $t = 2$ s

$$\vec{v}(2sec) = 2\hat{i} + 12\hat{j}$$

The speed of the particle at $t = 2$ s

$$v(2\,s) = \sqrt{2^2 + 12^2} = \sqrt{4 + 144}$$
$$= \sqrt{148} \approx 12.16\,m\,s^{-1}$$

Note that the particle has velocity components along x and y direction. Along the z direction the position has constant value (–5) which is independent of time. Hence there is no z-component for the velocity.

EXAMPLE 2.22

The velocity of three particles A, B, C are given below. Which particle travels at the greatest speed?

$$\vec{v_A} = 3\hat{i} - 5\hat{j} + 2\hat{k}$$
$$\vec{v_B} = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\vec{v_C} = 5\hat{i} + 3\hat{j} + 4\hat{k}$$

Solution

We know that speed is the magnitude of the velocity vector. Hence,

$$\text{Speed of A} = \left|\vec{v_A}\right| = \sqrt{(3)^2 + (-5)^2 + (2)^2}$$
$$= \sqrt{9 + 25 + 4} = \sqrt{38}\ \text{m s}^{-1}$$

$$\text{Speed of B} = \left|\vec{v_B}\right| = \sqrt{(1)^2 + (2)^2 + (3)^2}$$
$$= \sqrt{1 + 4 + 9} = \sqrt{14}\ \text{m s}^{-1}$$

$$\text{Speed of C} = \left|\vec{v_C}\right| = \sqrt{(5)^2 + (3)^2 + (4)^2}$$
$$= \sqrt{25 + 9 + 16} = \sqrt{50}\ \text{m s}^{-1}$$

The particle C has the greatest speed.

$$\sqrt{50} > \sqrt{38} > \sqrt{14}$$

EXAMPLE 2.23

Two cars are travelling with respective velocities $\vec{v}_1 = 10\,m\,s^{-1}$ along east and $\vec{v}_2 = 10\,m\,s^{-1}$ along west. What are the speeds of the cars?

Solution

Both cars have the same magnitude of velocity. This implies that both cars travel at the same speed even though they have velocities in different directions. Speed will not give the direction of motion.

Speedometer

Momentum

The linear momentum or simply momentum of a particle is defined as product of mass with velocity. It is denoted as '$\vec{p}$'. Momentum is also a vector quantity.

$$\vec{p} = m\vec{v}.$$

The direction of momentum is also in the direction of velocity, and the magnitude of momentum is equal to product of mass and speed of the particle.

$$p = mv$$

In component form the momentum can be written as

$$p_x\hat{i} + p_y\hat{j} + p_z\hat{k} = mv_x\hat{i} + mv_y\hat{j} + mv_z\hat{k}$$

Here $p_x = x$ component of momentum and is equal to mv_x

$p_y = y$ component of momentum and is equal to mv_y

$p_z = z$ component of momentum and is equal to mv_z

The momentum of the particle plays a very important role in Newton's laws. The physical significance of momentum can be well understood by the following example.

Consider a butterfly and a stone, both moving towards you with the same velocity 5 m s^{-1}. If both hit your body, the effects will not be the same. The effects not only depend upon the velocity, but also on the mass. The stone has greater mass compared to the butterfly. The momentum of the stone is thus greater than the momentum of the butterfly. It is the momentum which plays a major role in explaining the 'state' of motion of the object.

The unit of the momentum is kg m s^{-1}

EXAMPLE 2.24

Consider two masses of 10 g and 1 kg moving with the same speed 10 m s^{-1}. Calculate the magnitude of the momentum.

Solution

We use p = mv

For the mass of 10 g, m = 0.01 kg

$$p = 0.01 \times 10 = 0.1 \text{ kg m s}^{-1}$$

For the mass of 1 kg

$$p = 1 \times 10 = 10 \text{ kg m s}^{-1}$$

Thus even though both the masses have the same speed, the momentum of the heavier mass is 100 times greater than that of the lighter mass.

2.10 MOTION ALONG ONE DIMENSION

2.10.1 Average velocity

If a particle moves in one dimension, say for example along the x direction, then

$$\text{The average velocity} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}.$$

The average velocity is also a vector quantity. But in one dimension we have only two directions (positive and negative x direction), hence we use positive and negative signs to denote the direction.

The instantaneous velocity or velocity is defined as $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

Graphically the slope of the position-time graph will give the velocity of the particle. At the same time, if velocity time graph is given, the distance and displacement are determined by calculating the area under the curve. This is explained below.

We know that velocity is given by $\frac{dx}{dt} = v$.

Therefore, we can write $dx = vdt$

By integrating both sides, we get $\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt$.

As already seen, integration is equivalent to area under the given curve. So the term $\int_{t_1}^{t_2} v\,dt$ represents the area under the curve v as a function of time.

Since the left hand side of the integration represents the displacement travelled by the particle from time t_1 to t_2, the area under the velocity time graph will give the displacement of the particle. If the area is negative, it means that displacement is negative, so the particle has travelled in the negative direction. This is shown in the Figure 2.35 below.

EXAMPLE 2.25

A particle moves along the *x-axis* in such a way that its coordinates x varies with time 't' according to the equation $x = 2 - 5t + 6t^2$. What is the initial velocity of the particle?

Solution

$$x = 2 - 5t + 6t^2$$

$$\text{Velocity, } v = \frac{dx}{dt} = \frac{d}{dt}(2 - 5t + 6t^2)$$

$$\text{or } v = -5 + 12t$$

$$\text{For initial velocity, } t = 0$$

$$\therefore \text{ Initial velocity} = -5\text{ms}^{-1}$$

The negative sign implies that at t = 0 the velocity of the particle is along negative x direction.

Average speed = total path length / total time period

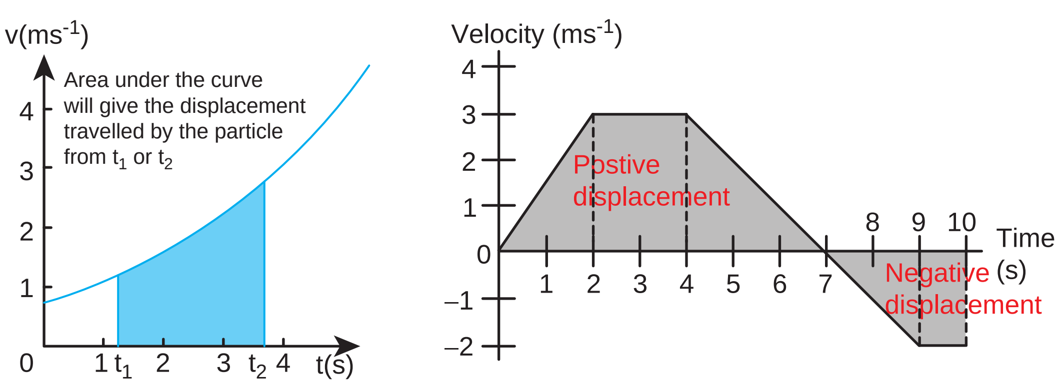


Figure 2.35 Displacement in the velocity–time graph

2.10.2 Relative Velocity in One and Two Dimensional Motion

When two objects A and B are moving with different velocities, then the velocity of one object A with respect to another object B is called relative velocity of object A with respect to B.

Case 1

Consider two objects A and B moving with uniform velocities V_A and $V_{B,}$ as shown, along straight tracks in the same direction $\vec{V}_A$, $\vec{V}_B$ with respect to ground.

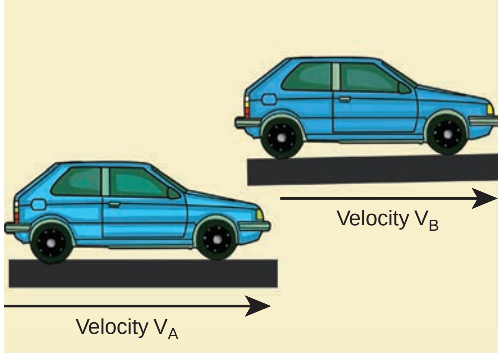
The relative velocity of object A with respect to object B is $\vec{V}_{AB} = \vec{V}_A - \vec{V}_B$

The relative velocity of object B with respect to object A is $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

Thus, if two objects are moving in the same direction, the magnitude of relative velocity of one object with respect to another is equal to the difference in magnitude of two velocities.

EXAMPLE 2.26

Suppose two cars A and B are moving with uniform velocities with respect to ground along parallel tracks and in the same direction. Let the velocities of A and B be 35 km h^{-1} due east and 40 km h^{-1} due east respectively. What is the relative velocity of car B with respect to A?



Solution

The relative velocity of B with respect to A, $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 5$ km h^{-1} due east

Similarly, the relative velocity of A with respect to B i.e., $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = 5$ km h^{-1} due west.

To a passenger in the car A, the car B will appear to be moving east with a velocity 5 km h^{-1}. To a passenger in car B, the car A will appear to move westwards with a velocity of 5 km h^{-1}

Case 2

Consider two objects A and B moving with uniform velocities V_A and V_B along the same straight tracks but opposite in direction

$$\xrightarrow[\vec{V}_A]{} \qquad \xleftarrow[\vec{V}_B]{}$$

The relative velocity of object A *with respect to* object B is

$$\vec{V}_{AB} = \vec{V}_A - (-\vec{V}_B) = \vec{V}_A + \vec{V}_B$$

The relative velocity of object B with respect to object A is

$$\vec{V}_{BA} = -\vec{V}_B - \vec{V}_A = -(\vec{V}_A + \vec{V}_B)$$

Thus, if two objects are moving in opposite directions, the magnitude of relative velocity of one object with respect to other is equal to the sum of magnitude of their velocities.

Case 3

Consider the velocities $\vec{v}_A$ and $\vec{v}_B$ at an angle θ between their directions.

The relative velocity of A with respect to B, $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$

Then, the magnitude and direction of $\vec{v}_{AB}$ is given by $v_{AB} = \sqrt{v_A^2 + v_B^2 - 2\, v_A v_B \cos\theta}$

and $\tan\beta = \dfrac{v_B \sin\theta}{v_A - v_B\cos\theta}$ (Here β is angle between $\vec{v}_{AB}$ and $\vec{v}_B$)

(i) When θ = 0°, the bodies move along parallel straight lines in the same direction,

We have $v_{AB} = (v_A - v_B)$ in the direction of $\vec{v}_A$. Obviously $v_{BA} = (v_B - v_A)$ in the direction of $\vec{v}_B$.

(ii) When θ = 180°, the bodies move along parallel straight lines in opposite directions,

We have $v_{AB} = (v_A + v_B)$ in the direction of $\vec{v}_A$.

Similarly, $v_{BA} = (v_B + v_A)$ in the direction of $\vec{v}_B$.

(iii) If the two bodies are moving at right angles to each other, then θ = 90°. The magnitude of the relative velocity of A with respect to $B = v_{AB} = \sqrt{v_A^2 + v_B^2}$.

(iv) Consider a person moving horizontally with velocity $\vec{V}_M$. Let rain fall vertically with velocity $\vec{V}_R$. An umbrella is held to avoid the rain. Then the relative velocity of the rain with respect to the person is, **(Figure 2.36)**

$$\vec{V}_{RM} = \vec{V}_R - \vec{V}_M$$

which has magnitude

$$V_{RM} = \sqrt{V_R^2 + V_M^2}$$

and direction $\theta = \tan^{-1}\left(\dfrac{V_M}{V_R}\right)$ with the vertical as shown in Figure. 2.36

In order to save himself from the rain, he should hold an umbrella at an angle θ with the vertical.

EXAMPLE 2.27

Suppose two trains A and B are moving with uniform velocities along parallel tracks but in opposite directions. Let the velocity of train A be 40 km h^{-1} due east and that of train B be 40 km h^{-1} due west. Calculate the relative velocities of the trains

Solution

Relative velocity of A with respect to B, v_{AB} = 80 km h^{-1} due east

Thus to a passenger in train B, the train A will appear to move east with a velocity of 80 km h^{-1}

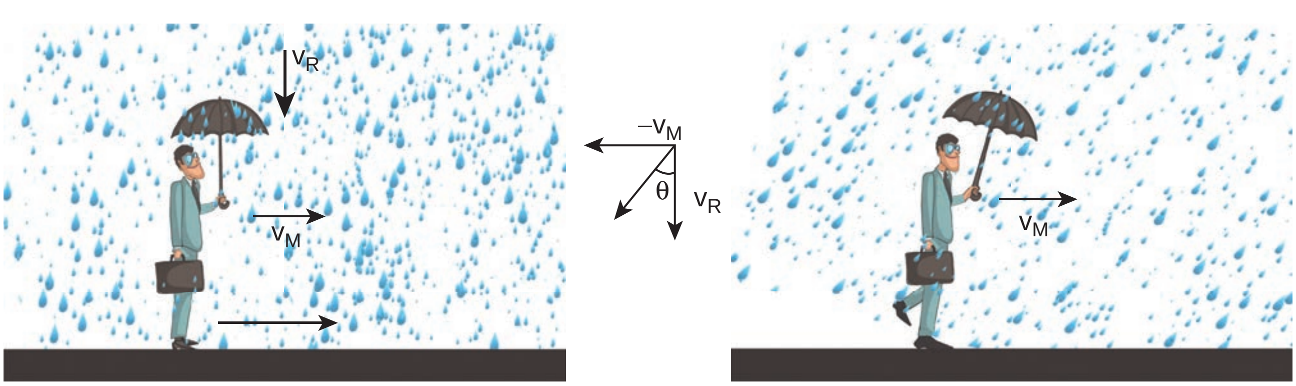


Figure 2.36 Angle of umbrella with respect to rain

The relative velocity of B with respect to A, $V_{BA} = 80 \text{ km h}^{-1}$ due west

To a passenger in train A, the train B will appear to move westwards with a velocity of 80 km h^{-1}

EXAMPLE 2.28

Consider two trains A and B moving along parallel tracks with the same velocity in the same direction. Let the velocity of each train be 50 km h^{-1} due east. Calculate the relative velocities of the trains.

Solution

Relative velocity of B with respect to A,

$$v_{BA} = v_B - v_A$$
$$= 50 \text{ km h}^{-1} + (-50) \text{ km h}^{-1}$$
$$= 0 \text{ km h}^{-1}$$

Similarly, relative velocity of A with respect to B i.e., v_{AB} is also zero.

Thus each train will appear to be at rest with respect to the other.

EXAMPLE 2.29

How long will a boy sitting near the window of a train travelling at 36 km h^{-1} see a train passing by in the opposite direction with a speed of 18 km h^{-1}. The length of the slow-moving train is 90 m.

Solution

The relative velocity of the slow-moving train with respect to the boy is = (36 + 18) $\text{km h}^{-1} = 54 \text{ km h}^{-1} = 54 \times \frac{5}{18} \text{ m s}^{-1} = 15 \text{ m s}^{-1}$

Since the boy will watch the full length of the other train, to find the time taken to watch the full train:

We have, $15 = \frac{90}{t}$

or $t = \frac{90}{15} = 6\,\text{s}$

EXAMPLE 2.30

A swimmer's speed in the direction of flow of a river is 12 km h^{-1}. Against the direction of flow of the river the swimmer's speed is 6 km h^{-1}. Calculate the swimmer's speed in still water and the velocity of the river flow.

Solution

Let v_s and v_r, represent the velocities of the swimmer and river respectively with respect to ground.

$$v_s + v_r = 12 \quad (1)$$

and $$v_s - v_r = 6 \quad (2)$$

Adding the both equations (1) and (2)

$2v_s = 12 + 6 = 18 \text{ km h}^{-1}$ or $v_s = 9 \text{ km h}^{-1}$

From Equation (1),

$9 + v_r = 12$ or

$v_r = 3 \text{ km h}^{-1}$

When the river flow and swimmer move in the same direction, the net velocity of swimmer is 12 km h^{-1}.

Accelerated Motion

During non-uniform motion of an object, the velocity of the object changes from instant to instant i.e., the velocity of the object is no more constant but changes with time. Such a motion is said to be an accelerated motion.

i) In accelerated motion, if the change in velocity of an object per unit time is same (constant) then the object is said to be moving with uniformly accelerated motion.
ii) On the other hand, if the change in velocity per unit time is different at different times, then the object is said to be moving with non-uniform accelerated motion.

Average acceleration

If an object changes its velocity from $\vec{v}_1$ to $\vec{v}_2$ in a time interval $\Delta t = t_2 - t_1$, then the average acceleration is defined as the ratio of change in velocity over the time interval $\Delta t = t_2 - t_1$

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

Average acceleration is a vector quantity in the same direction as the vector $\Delta \vec{v}$.

Instantaneous acceleration

Usually, the average acceleration will give the change in velocity only over the entire time interval. It will not give value of the acceleration at any instant time t.

Instantaneous acceleration or acceleration of a particle at time 't' is given by the ratio of change in velocity over Δt, as Δt approaches zero.

$$\text{Acceleration } \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

In other words, the acceleration of the particle at an instant t is equal to rate of change of velocity.

(i) Acceleration is a vector quantity. Its SI unit is ms^{-2} and its dimensional formula is $M^0L^1T^{-2}$
(ii) Acceleration is positive if its velocity is increasing, and is negative if the velocity is decreasing. The negative acceleration is called retardation or deceleration.

In terms of components, we can write

$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = \frac{d\vec{v}}{dt}$$

Thus $a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt}, a_z = \frac{dv_z}{dt}$ are the components of instantaneous acceleration.

Since each component of velocity is the derivative of the corresponding coordinate, we can express the components a_x, a_y, and a_z, as

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}, \quad a_z = \frac{d^2z}{dt^2}$$

Then the acceleration vector $\vec{a}$ itself is

$$\vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} = \frac{d^2\vec{r}}{dt^2}$$

Thus acceleration is the second derivative of position vector with respect to time.

Graphically the acceleration is the slope in the velocity-time graph. At the same time if the acceleration-time graph is given, then the velocity can be found from the area under the acceleration-time graph.

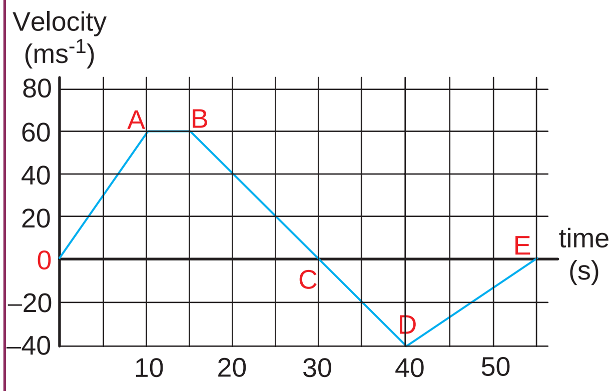
From $\frac{dv}{dt} = a$, we have $dv = a\,dt$; hence

$$v = \int_{t_1}^{t_2} a\,dt$$

For an initial time t_1 and final time t_2

EXAMPLE 2.31

A velocity–time graph is given for a particle moving in x direction, as below



a) Describe the motion qualitatively in the interval 0 to 55 s.

b) Find the distance and displacement travelled from 0 s to 40 s.

c) Find the acceleration at t = 5 s and at t = 20 s

Solution

(a) **From O to A: (0 s to 10 s)**

At t = 0 s the particle has zero velocity. At t > 0, particle has positive velocity and moves in the positive x direction. From 0 s to 10 s the slope $(\frac{dv}{dt})$ is positive, implying the particle is accelerating. Thus the velocity increases during this time interval.

From A to B: (10 s to 15 s)

From 10 s to 15 s the velocity stays constant at 60 m s^{-1}. The acceleration is 0 during this period. But the particle continues to travel in the positive *x*-direction.

From B to C : (15 s to 30 s)

From the 15 s to 30 s the slope is negative, implying the velocity is decreasing. But the particle is moving in the positive x direction. At t = 30 s the velocity becomes zero, and the particle comes to rest momentarily at t = 30 s.

From C to D: (30 s to 40 s)

From 30 s to 40 s the velocity is negative. It implies that the particle starts to move in the negative x direction. The magnitude of velocity increases to a maximum 40 m s^{-1}

From D to E: (40 s to 55 s)

From 40 s to 55 s the velocity is still negative, but starts increasing from –40 m s^{-1} At t = 55 s the velocity of the particle is zero and particle comes to rest.

(b) The total area under the curve from 0 s to 40 s will give the displacement. Here the area from O to C represents motion along positive x–direction and the area under the graph from C to D represents the particle's motion along negative x–direction.

The displacement travelled by the particle from 0 s to 10 s = $\frac{1}{2}\times 10\times 60 = 300\,m$

The displacement travelled from 10 s to 15 s = 60 × 5 = 300 m

The displacement travelled from 15 s to 30 s = $\frac{1}{2}\times 15\times 60 = 450\,m$

The displacement travelled from 30 s to 40 s = $\frac{1}{2}\times 10\times(-40) = -200\,m.$

Here the negative sign implies that the particle travels 200 m in the negative x direction.

The total displacement from 0 s to 40 s is given by

300 m + 300 m + 450 m – 200 m
= +850 m.

Thus the particle's net displacement is along the positive x-direction.

The total distance travelled by the particle from 0 s to 40 s = 300 + 300 + 450 + 200 = 1250 m.

(c) The acceleration is given by the slope in the velocity-time graph. In the first 10 seconds the velocity has constant slope (constant acceleration). It implies that the acceleration a is from $v_1 = 0$ to $v_2 = 60\,\text{m s}^{-1}$.

Hence $a = \dfrac{v_2 - v_1}{t_2 - t_1}$ gives

$$a = \frac{60-0}{10-0} = 6\,m\,s^{-2}$$

Next, the particle has constant negative slope from 15 s to 30 s. In this case $v_2 = 0$ and $v_1 = 60\,m\,s^{-1}$. Thus the acceleration at t = 20 s is given by $a = \dfrac{0-60}{30-15} = -4\,m\,s^{-2}$. Here the negative sign implies that the particle has negative acceleration.

EXAMPLE 2.32

If the position vector of the particle is given by $\vec{r} = 3t^2\hat{i} + 5t\hat{j} + 4\hat{k}$, Find the

a) The velocity of the particle at t = 3 s
b) Speed of the particle at t = 3 s
c) acceleration of the particle at time t = 3 s

Solution

(a) The velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$

We obtain, $\vec{v}(t) = 6t\hat{i} + 5\hat{j}$

The velocity has only two components $v_x = 6t$, depending on time t and $v_y = 5$ which is independent of time.

The velocity at t = 3 s is $\vec{v}(3) = 18\hat{i} + 5\hat{j}$

(b) The speed at t = 3 s is $v = \sqrt{18^2 + 5^2} = \sqrt{349} \approx 18.68\,m\,s^{-1}$

(c) The acceleration $\vec{a}$ is, $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 6\hat{i}$

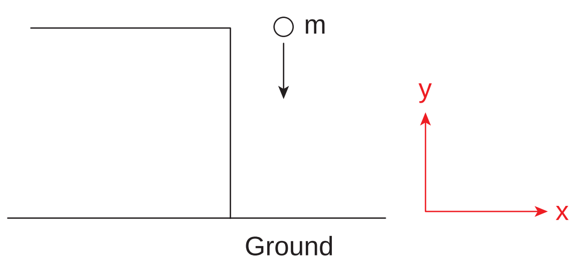
The acceleration has only the x-component. Note that acceleration here is independent of t, which means $\vec{a}$ is constant. Even at t = 3 s it has same value $\vec{a} = 6\hat{i}$. The velocity is non-uniform, but the acceleration is uniform (constant) in this case.

EXAMPLE 2.33

An object is thrown vertically downward. What is the acceleration experienced by the object?

Solution

We know that when the object falls towards the Earth, it experiences acceleration due to gravity g = 9.8 m s^{-2} downward. We can choose the coordinate system as shown in the figure.



The acceleration is along the negative y direction.

$$\vec{a} = g\left(-\hat{j}\right) = -g\hat{j}$$

Note For convenience, sometimes we take the downward direction as positive Y-axis. As a vertically falling body accelerates downwards, g is taken as positive in this direction. (a = g)

2.10.3 Equations of Uniformly Accelerated Motion by Calculus Method

Consider an object moving in a straight line with uniform or constant acceleration 'a'.

Let u be the velocity of the object at time t = 0, and v be velocity of the body at a later time t.

Velocity - time relation

(i) The acceleration of the body at any instant is given by the first derivative of the velocity with respect to time,

$$a = \frac{dv}{dt} \text{ or } dv = a\,dt$$

Integrating both sides with the condition that as time changes from 0 to t, the velocity changes from u to v. For the constant acceleration,

$$\int_u^v dv = \int_0^t a\,dt = a\int_0^t dt \Rightarrow \left[v\right]_u^v = a\left[t\right]_0^t$$

$$v - u = at \quad (or) \quad v = u + at \qquad \rightarrow (2.7)$$

Note If *a* is dependent on time *t*, *a* cannot be brought outside the time integral.

Displacement – time relation

(ii) The velocity of the body is given by the first derivative of the displacement with respect to time.

$$v = \frac{ds}{dt} \text{ or } ds = vdt$$

and since $v = u + at$,

We get $ds = \left(u + at\right)dt$

Assume that initially at time t = 0, the particle started from the origin. At a later time t, the particle displacement is s. Further assuming that acceleration is time-independent, we have

$$\int_0^s ds = \int_0^t u\,dt + \int_0^t at\,dt \;\left(or\right)\; s = ut + \frac{1}{2}at^2 \quad (2.8)$$

Velocity – displacement relation

(iii) The acceleration is given by the first derivative of velocity with respect to time.

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

[since ds/dt = v] where s is displacement traversed.

This is rewritten as $a = \frac{1}{2}\frac{d}{ds}(v^2)$

or $ds = \frac{1}{2a}d(v^2)$

Integrating the above equation, using the fact when the velocity changes from u to v, displacement changes from 0 to s, we get

$$\int_0^s ds = \int_u^v \frac{1}{2a}d\left(v^2\right)$$

$$\therefore s = \frac{1}{2a}\left(v^2 - u^2\right)$$

$$\therefore v^2 = u^2 + 2as \qquad (2.9)$$

We can also derive the displacement s in terms of initial velocity u and final velocity v.

From the equation (2.7) we can write,

$$at = v - u$$

Substitute this in equation (2.8), we get

$$s = ut + \frac{1}{2}(v-u)t$$

$$s = \frac{(u+v)t}{2} \qquad (2.10)$$

The equations (2.7), (2.8), (2.9) and (2.10) are called kinematic equations of motion, and have a wide variety of practical applications.

Kinematic equations

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{(u+v)t}{2}$$

It is to be noted that all these kinematic equations are valid only if the motion is in a straight line with constant acceleration. For circular motion and oscillatory motion these equations are not applicable.

Equations of motion under gravity

A practical example of a straight line motion with constant acceleration is the motion of an object near the surface of the Earth. We know that near the surface of the Earth, the acceleration due to gravity 'g' is constant. All straight line motions under this acceleration can be well understood using the kinematic equations given earlier.

Case (i): A body falling from a height h

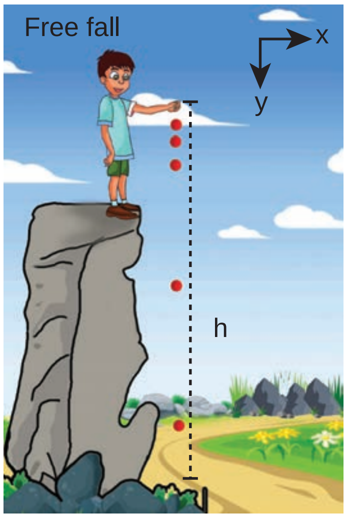


Figure 2.37 An object in free fall

Consider an object of mass m falling from a height h. Assume there is no air resistance. For convenience, let us choose the downward direction as positive y-axis as shown in the Figure 2.37. The object experiences acceleration 'g' due to gravity which is constant near the surface of the Earth. We can use kinematic equations to explain its motion. We have

The acceleration $\vec{a} = g\hat{j}$

By comparing the components, we get

$$a_x = 0, a_z = 0, a_y = g$$

Let us take for simplicity, $a_y = a = g$

If the particle is thrown with initial velocity 'u' downward which is in negative y axis, then velocity and position at of the particle any time t is given by

$$v = u + gt \quad (2.11)$$

$$y = ut + \frac{1}{2}gt^2 \quad (2.12)$$

The square of the speed of the particle when it is at a distance y from the hill-top, is

$$v^2 = u^2 + 2gy \quad (2.13)$$

Suppose the particle starts from rest.

Then u = 0

Then the velocity v, the position of the particle and v^2 at any time t are given by (for a point y from the hill-top)

$$v = gt \quad (2.14)$$

$$y = \frac{1}{2}gt^2 \quad (2.15)$$

$$v^2 = 2gy \quad (2.16)$$

The time (t = T) taken by the particle to reach the ground (for which y = h), is given by using equation (2.15),

$$h = \frac{1}{2}gT^2 \quad (2.17)$$

$$T = \sqrt{\frac{2h}{g}} \quad (2.18)$$

The equation (2.18) implies that greater the height(h), particle takes more time(T) to reach the ground. For lesser height(h), it takes lesser time to reach the ground.

The speed of the particle when it reaches the ground (y = h) can be found using equation (2.16), we get

$$v_{ground} = \sqrt{2gh} \quad (2.19)$$

The above equation implies that the body falling from greater height(h) will have higher velocity when it reaches the ground.

The motion of a body falling towards the Earth from a small altitude (h << R), purely under the force of gravity is called free fall. (Here R is radius of the Earth)

EXAMPLE 2.34

An iron ball and a feather are both falling from a height of 10 m.

a) What are the time taken by the iron ball and feather to reach the ground?

b) What are the velocities of iron ball and feather when they reach the ground? (Ignore air resistance and take $g = 10\ \text{m s}^{-2}$)

Solution

Since kinematic equations are independent of mass of the object, according to equation (2.8) the time taken by both iron ball and feather to reach the ground are the same. This is given by

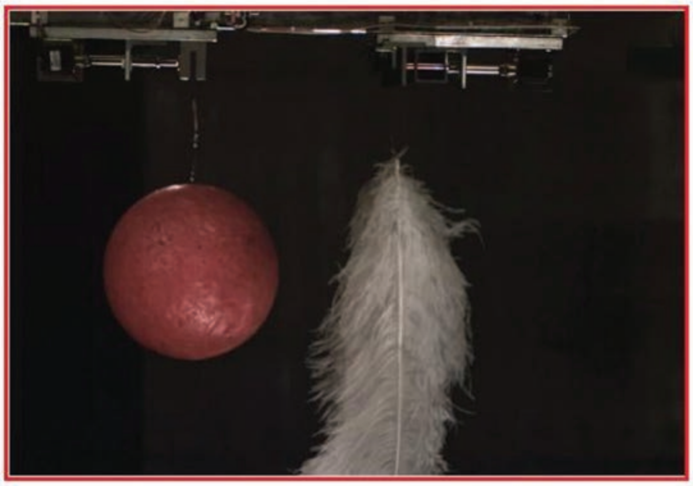
$$T = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 10}{10}} = \sqrt{2}\,s \approx 1.414\,s$$

Thus, both feather and iron ball reach ground at the same time.

By following equation (2.19) both iron ball and feather reach the Earth with the same speed. It is given by

$$v = \sqrt{2gh} = \sqrt{2 \times 10 \times 10}$$
$$= \sqrt{200}\, m\, s^{-1} \approx 14.14\, m\, s^{-1}$$

Galileo concluded that in vaccum all objects fall with the same acceleration g and reach the ground at the same time.



https://www.youtube.com/watch?v=E43-CfukEgs&t=5s

EXAMPLE 2.35

Is it possible to measure the depth of a well using kinematic equations?

Consider a well without water, of some depth d. Take a small object (for example lemon) and a stopwatch. When you drop the lemon, start the stop watch. As soon as the lemon touches the bottom of the well, stop the watch. Note the time taken by the lemon to reach the bottom and denote the time as t.

Since the initial velocity of lemon $u = 0$ and the acceleration due to gravity g is constant over the well, we can use the equations of motion for constant acceleration.

$$s = ut + \frac{1}{2}at^2$$

Since $u = 0, s = d, a = g$ (Since we choose the y axis downwards), Then

$$d = \frac{1}{2} g t^2$$

Substituting $g = 9.8 \text{ m s}^{-2}$ we get the depth of the well.

To estimate the error in our calculation we can use another method to measure the depth of the well. Take a long rope and hang the rope inside the well till it touches the bottom. Measure the length of the rope which is the correct depth of the well ($d_{correct}$). Then

$$error = d_{correct} - d$$

$$relative\ error = \frac{d_{correct} - d}{d_{correct}}$$

$$percentage\ of\ relative\ error = \frac{d_{correct} - d}{d_{correct}} \times 100$$

What would be the reason for an error, if any?

Repeat the experiment for different masses and compare the result with $d_{correct}$ every time.

Note If there is water in the well, this method can be used to measure depth of the well till the surface of the water.

Case (ii): A body thrown vertically upwards

Consider an object of mass m thrown vertically upwards with an initial velocity u. Let us neglect the air friction. In this case we choose the vertical direction as positive y axis as shown in the Figure 2.38, then the acceleration a = −g (neglect air friction) and g points towards the negative y axis. The kinematic equations for this motion are,

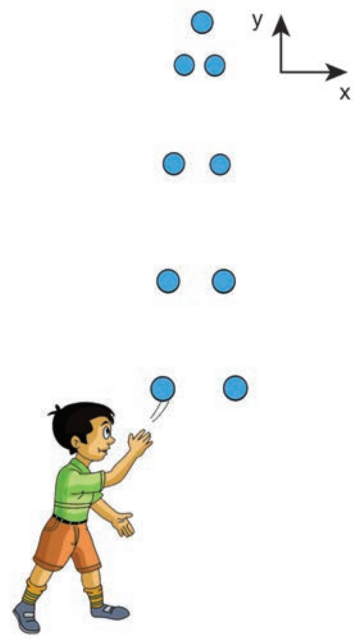


Figure 2.38. An object thrown vertically

The velocity and position of the object at any time t are,

$$v = u - gt \qquad (2.20)$$

$$s = ut - \frac{1}{2} g t^2 \qquad (2.21)$$

The velocity of the object at any position y (from the point where the object is thrown) is

$$v^2 = u^2 - 2gy \qquad (2.22)$$

EXAMPLE 2.36

A train was moving at the rate of 54 km h^{-1} when brakes were applied. It came to rest within a distance of 225 m. Calculate the retardation produced in the train.

Solution

The final velocity of the particle $v = 0$

The initial velocity of the particle

$$u = 54 \times \frac{5}{18} m\,s^{-1} = 15\ m\,s^{-1}$$

$$S = 225\ m$$

Retardation is always against the velocity of the particle.

$$v^2 = u^2 - 2aS$$

$$0 = (15)^2 - 2a\,(225)$$

$$450\ a = 225$$

$$a = \frac{225}{450} ms^{-2} = 0.5 ms^{-2}$$

Hence, retardation $= 0.5\ m\ s^{-2}$

2.11 PROJECTILE MOTION

2.11.1 Introduction

When an object is thrown in the air with some initial velocity (NOT just upwards), and then allowed to move under the action of gravity alone, the object is known as a projectile. The path followed by the particle is called its **trajectory.**

Examples of projectile are

1. An object dropped from window of a moving train.
2. A bullet fired from a rifle.
3. A ball thrown in any direction.
4. A javelin or shot put thrown by an athlete.
5. A jet of water issuing from a hole near the bottom of a water tank.

It is found that a projectile moves under the combined effect of two velocities.

i) A uniform velocity in the horizontal direction, which will not change provided there is no air resistance.
ii) **A uniformly changing velocity (i.e., increasing or decreasing) in the vertical direction.**

There are two types of projectile motion:

(i) Projectile given an initial velocity in the horizontal direction (horizontal projection)
(ii) Projectile given an initial velocity at an angle to the horizontal (angular projection)

To study the motion of a projectile, let us assume that,

i) Air resistance is neglected.
ii) The effect due to rotation of Earth and curvature of Earth is negligible.
iii) The acceleration due to gravity is constant in magnitude and direction at all points of the motion of the projectile.

2.11.2 Projectile in horizontal projection

Consider a projectile, say a ball, thrown horizontally with an initial velocity $\vec{u}$ from the top of a tower of height h (Figure 2.39).

As the ball moves, it covers a horizontal distance due to its uniform horizontal

UNIT-2(XI-Physics_Vol-1)-1.indd 81 05-01-2022 22:15:27

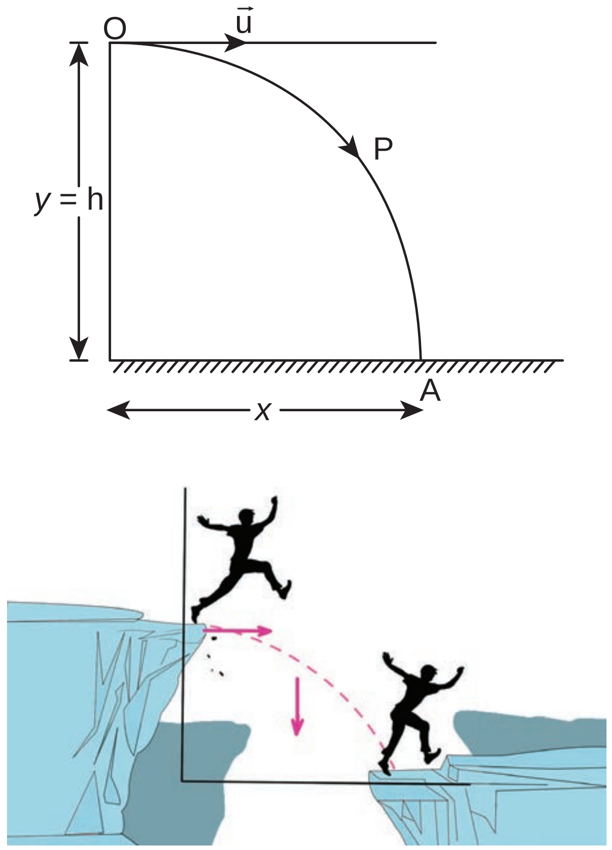


Figure 2.39 Horizontal Projection.

velocity u, and a vertical downward distance because of constant acceleration due to gravity g. Thus, under the combined effect the ball moves along the path OPA. The motion is in a 2-dimensional plane. Let the ball take time t to reach the ground at point A, Then the horizontal distance travelled by the ball is x (t) = x, and the vertical distance travelled is y (t) = y

We can apply the kinematic equations along the x direction and y direction separately. Since this is two-dimensional motion, the velocity will have both horizontal component u_x and vertical component u_y.

Motion along horizontal direction

The particle has zero acceleration along x direction. So, the initial velocity u_x remains constant throughout the motion.

The distance traveled by the projectile at a time t is given by the equation $x = u_x t + \frac{1}{2}at^2$. Since a = 0 along x direction, we have

$$x = u_x t \qquad (2.23)$$

Motion along downward direction

Here u_y = 0 (initial velocity has no downward component), a = g (we choose the +ve y-axis in downward direction), and distance y at time t

$\therefore$ From equation, $y = u_y t + \frac{1}{2}at^2$, we get

$$y = \frac{1}{2}gt^2 \qquad (2.24)$$

Substituting the value of t from equation (2.23) in equation (2.24) we have

$$y = \frac{1}{2}g\frac{x^2}{u_x^2} = \left(\frac{g}{2u_x^2}\right)x^2$$

$$y = Kx^2 \qquad (2.25)$$

where $K = \frac{g}{2u_x^2}$ is constant

Equation (2.25) is the equation of a parabola. Thus, the path followed by the projectile is a parabola (curve OPA in the Figure 2.39).

(1) Time of Flight: *The time taken for the projectile to complete its trajectory or time taken by the projectile to hit the ground is called time of flight.*

Consider the example of a tower and projectile. Let h be the height of a tower. Let T be the time taken by the projectile to hit the ground, after being thrown horizontally from the tower.

UNIT-2(XI-Physics_Vol-1)-1.indd 82 05-01-2022 22:15:29

We know that $s_y = u_y t + \frac{1}{2}at^2$ for vertical motion. Here $s_y = h$, $t = T$, $u_y = 0$ (i.e., no initial vertical velocity). Then

$$h = \frac{1}{2}gT^2 \quad \text{or} \quad T = \sqrt{\frac{2h}{g}}$$

Thus, the time of flight for projectile motion depends on the height of the tower, but is independent of the horizontal velocity of projection. If one ball falls vertically and another ball is projected horizontally with some velocity, both the balls will reach the bottom at the same time. This is illustrated in the Figure 2.40.

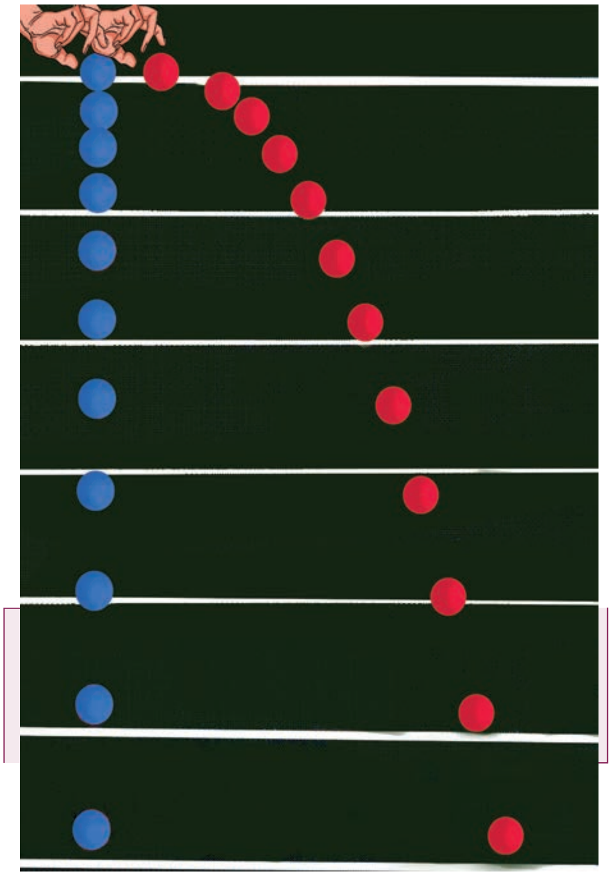


Figure 2.40 Vertical distance covered by the two particles is same in equal intervals.

(2) Horizontal range: The horizontal distance covered by the projectile from the foot of the tower to the point where the projectile hits the ground is called *horizontal range*. For horizontal motion, we have

$$s_x = u_x t + \frac{1}{2}at^2.$$

Here, s_x = R (range), $u_x = u$, a = 0 (no horizontal acceleration) T is time of flight. Then horizontal range = uT.

Since the time of flight $T = \sqrt{\frac{2h}{g}}$, we substitute this and we get the horizontal range of the particle as $R = u\sqrt{\frac{2h}{g}}$

The above equation implies that the range R is directly proportional to the initial velocity u and inversely proportional to acceleration due to gravity g.

(3) Resultant Velocity (Velocity of projectile at any time): At any instant t, the projectile has velocity components along both x-axis and y-axis. The resultant of these two components gives the velocity of the projectile at that instant t, as shown in Figure 2.41.

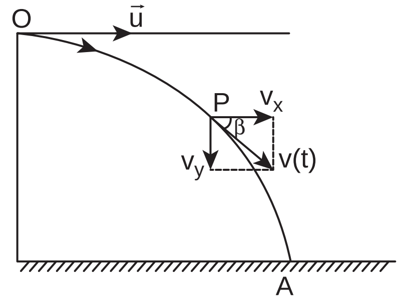


Figure 2.41. Velocity resolved into two components

The velocity component at any t along horizontal (x-axis) is $v_x = u_x + a_x t$

Since, $u_x = u$, $a_x = 0$, we get

$$v_x = u \qquad \rightarrow (2.26)$$

The component of velocity along vertical direction (y-axis) is $v_y = u_y + a_y t$

Since, $u_y = 0$, $a_y = g$, we get

$$v_y = gt \qquad \rightarrow (2.27)$$

Hence the velocity of the particle at any instant is

$$\vec{v} = u\hat{i} + gt\hat{j}$$

The speed of the particle at any instant t is given by

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2}$$
$$v = \sqrt{u^2 + g^2t^2}$$

(4) Speed of the projectile when it hits the ground: When the projectile hits the ground after initially thrown horizontally from the top of tower of height h, the time of flight is

$$T = \sqrt{\frac{2h}{g}}$$

The horizontal component velocity of the projectile remains the same i.e $v_x = u$

The vertical component velocity of the projectile at time T is

$$v_y = gT = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}$$

The speed of the particle when it reaches the ground is

$$v = \sqrt{u^2 + 2gh}$$

2.11.3 Projectile under an angular projection

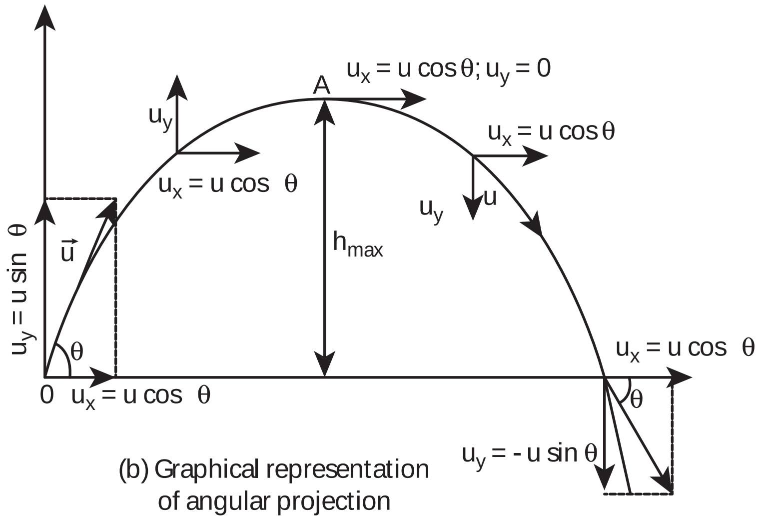
This projectile motion takes place when the initial velocity is not horizontal, but at some angle with the vertical, as shown in Figure 2.42.

(Oblique projectile)

Examples:

- Water ejected out of a hose pipe held obliquely.
- Cannon fired in a battle ground.

(a) Water is ejected out of a pipe held obliquely



(b) Graphical representation of angular projection

Figure 2.42. Projectile motion

Consider an object thrown with initial velocity $\vec{u}$ at an angle θ with the horizontal. Refer Figures 2.42 and 2.43.

Then,

$$\vec{u} = u_x\hat{i} + u_y\hat{j}$$

where $u_x = u\cos\theta$ is the horizontal component and $u_y = u\sin\theta$ the vertical component of velocity.

Since the acceleration due to gravity is in the direction opposite to the direction of vertical component u_y, this component will gradually reduce to zero at the maximum height of the projectile. At this maximum height, the same gravitational force will push the projectile to move downward and fall to the ground. There is no acceleration along the x direction throughout the motion. So, the horizontal component of the velocity ($u_x = u\cos\theta$) remains the same till the object reaches the ground.

Hence after the time t, the velocity along horizontal motion $v_x = u_x + a_x t = u_x = u\cos\theta$

The horizontal distance travelled by projectile in time t is $s_x = u_x t + \frac{1}{2}a_x t^2$

Here, $s_x = x$, $u_x = u\cos\theta$, $a_x = 0$

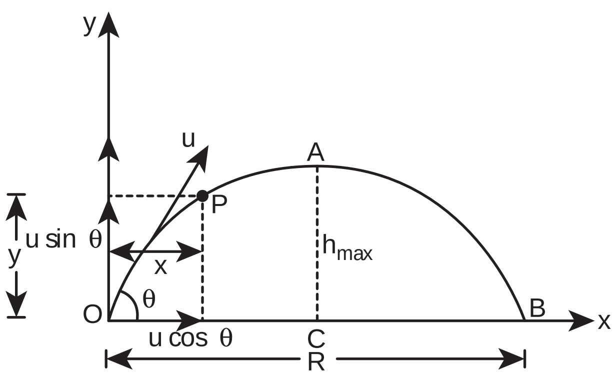


Figure 2.43. Initial velocity resolved into components

$$\text{Thus, } x = u\cos\theta . t \text{ or } t = \frac{x}{u\cos\theta} \quad (2.28)$$

Next, for the vertical motion $v_y = u_y + a_y t$

Here $u_y = u\sin\theta$, $a_y = -g$ (acceleration due to gravity acts opposite to the motion). Thus

$$\text{Thus, } v_y = u\sin\theta - gt \quad (2.29)$$

The vertical distance travelled by the projectile in the same time t is $s_y = u_y t + \frac{1}{2}a_y t^2$

Here, $s_y = y$, $u_y = u\sin\theta$, $a_y = -g$. Then

$$y = u\sin\theta\, t - \frac{1}{2}gt^2 \quad (2.30)$$

Substitute the value of t from equation (2.28) in equation (2.30), we have

$$y = u\sin\theta \frac{x}{u\cos\theta} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta} \quad (2.31)$$

Thus the path followed by the projectile is an inverted parabola.

Maximum height (h_{max})

The maximum vertical distance travelled by the projectile during its journey is called maximum height. This is determined as follows:

For the vertical part of the motion,

$$v_y^2 = u_y^2 + 2a_y s$$

Here, $u_y = u\sin\theta$, $a_y = -g$, $s = h_{max}$, and at the maximum height $v_y = 0$

Hence,

$$0 = u^2 \sin^2\theta - 2gh_{max}$$

$$\text{Or } h_{max} = \frac{u^2 \sin^2 \theta}{2g} \qquad (2.32)$$

Time of flight (T_f)

The total time taken by the projectile from the point of projection till it hits the horizontal plane is called time of flight.

This time of flight is the time taken by the projectile to go from point O to B via point A (Figure 2.43)

We know that $s_y = u_y t + \frac{1}{2} a_y t^2$

Here, $s_y = y = 0$ (net displacement in y-direction is zero), $u_y = u\,sin\theta$, $a_y = -g$, $t = T_f$

Then

$$0 = u \sin\theta\, T_f - \frac{1}{2} g T_f^2$$

$$T_f = 2u \frac{\sin\theta}{g}$$

Horizontal range (R)

The maximum horizontal distance between the point of projection and the point on the horizontal plane where the projectile hits the ground is called horizontal range (R). This is found easily since the horizontal component of initial velocity remains the same. We can write

Range R = Horizontal component of velocity x time of flight = $u \cos\theta \times T_f$

$$R = u \cos\theta \times \frac{2u \sin\theta}{g} = \frac{2u^2 \sin\theta \cos\theta}{g}$$

$$\therefore R = \frac{u^2 \sin 2\theta}{g} \qquad (2.33)$$

The horizontal range directly depends on the initial speed (u) and the sine of angle of projection (θ). It inversely depends on acceleration due to gravity 'g'

For a given initial speed u, the maximum possible range is reached when $\sin 2\theta$ is maximum, $\sin 2\theta = 1$. This implies $2\theta = \pi/2$

$$\text{or} \qquad \theta = \frac{\pi}{4}$$

This means that if the particle is projected at 45 degrees with respect to horizontal, it attains maximum range, given by.

$$R_{max} = \frac{u^2}{g} \qquad (2.34)$$

Projectile motion !

In Tamil Nadu there is an interesting traditional game 'kitti pull'. When the 'pull' is hit by the kitti, the path followed by the pull is 'parabolic'.

EXAMPLE 2.37

Suppose an object is thrown with initial speed 10 m s^{-1} at an angle $\pi/4$ with the horizontal, what is the range covered? Suppose the same object is thrown similarly in the Moon, will there be any change in the range? If yes, what is the change? (The acceleration due to gravity in the Moon $g_{moon} = \frac{1}{6}g$)

Solution

In projectile motion, the range of particle is given by,

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\theta = \pi/4 \quad u = v_0 = 10m\ s^{-1}$$

$$\therefore R_{earth} = \frac{(10)^2 \sin \pi/2}{9.8} = 100/9.8$$

$R_{earth} = 10.20\ m$ (Approximately 10 m)

If the same object is thrown in the Moon, the range will increase because in the Moon, the acceleration due to gravity is smaller than g on Earth,

$$g_{moon} = \frac{g}{6}$$

$$R_{moon} = \frac{u^2 \sin 2\theta}{g_{moon}} = \frac{v_0^2 \sin 2\theta}{g/6}$$

$$\therefore R_{moon} = 6R_{earth}$$

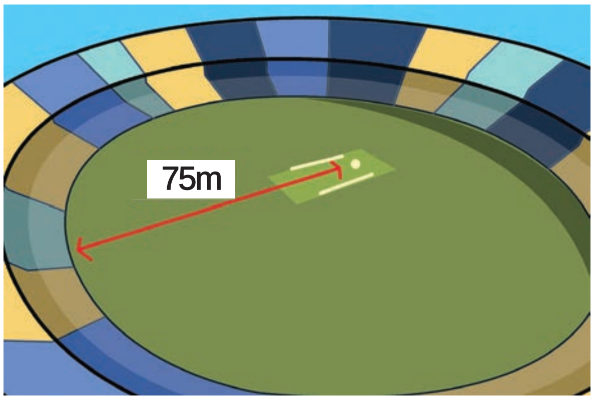
$$R_{moon} = 6 \times 10.20 = 61.20\ m$$

(Approximately 60 m)

The range attained on the Moon is approximately six times that on Earth.

EXAMPLE 2.38

In the cricket game, a batsman strikes the ball such that it moves with the speed 30 m s^{-1} at an angle 30^0 with the horizontal as shown in the figure. The boundary line of the cricket ground is located at a distance of 75 m from the batsman? Will the ball go for a six? (Neglect the air resistance and take acceleration due to gravity g = 10 m s^{-2}).



Solution

The motion of the cricket ball in air is essentially a projectile motion. As we have already seen, the range (horizontal distance) of the projectile motion is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

The initial speed $u = 30\,m\ s^{-1}$

The projection angle $\theta = 30°$

The horizontal distance travelled by the cricket ball

$$R = \frac{(30)^2 \times \sin 60°}{10} = \frac{900 \times \frac{\sqrt{3}}{2}}{10} = 77.94\,m$$

This distance is greater than the distance of the boundary line. Hence the ball will cross this line and go for a six.

2.11.4 Introduction to Degrees and Radians

In measuring angles, there are several possible units used, but the most common units are degrees and radians. Radians are used in measuring area, volume, and circumference of circles and surface area of spheres.

Radian describes the planar angle subtended by a circular arc at the centre of a circle. It is defined as the length of the arc divided by the radius of the arc. One radian is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle. This is shown in the Figure 2.44.

θ = 1 radian (rad)

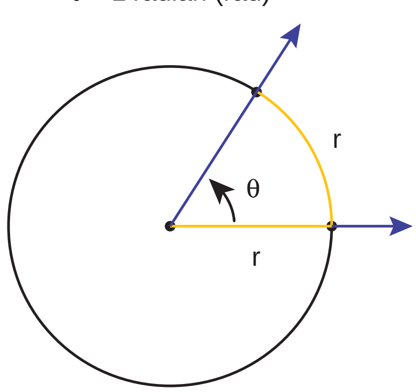


Figure 2.44 One radian (shown in yellow color)

Degree is the unit of measurement which is used to determine the size of an angle. When an angle goes all the way around in a circle, the total angle covered is equivalent to 360°. Thus, a circle has 360°. In terms of radians, the full circle has 2π radian.

Hence we write $360° = 2\pi$ *radians*

or $1 \text{ radians} = \frac{180}{\pi}$ degrees

which means $1 \text{ rad} \cong 57.27°$

EXAMPLE 2.39

Calculate the angle θ subtended by the two adjacent wooden spokes of a bullock cart wheel is shown in the figure. Express the angle in both radian and degree.

Solution

The full wheel subtends 2π radians at the centre of the wheel. The wheel is divided into 12 parts (arcs).

So one part subtends an angle $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ radian at the centre

Since, $\pi\ rad = 180°$, $\frac{\pi}{6}$ radian is equal to 30 degree.

∴ The angle subtended by two adjacent wooden spokes is 30 degree at the centre.

Note The number π plays a very important role in both physics and mathematics. It is an irrational number which cannot be expressed in finite decimal form. However, we can approximate π as ≈ 3.14 or $\frac{22}{7}$. But we should not forget that 3.14 or $\frac{22}{7}$ is only an approximation to π and not equal to π.

2.11.5 Angular displacement

Consider a particle revolving around a point O in a circle of radius r (Figure 2.45). Let the position of the particle at time t = 0 be A and after time t, its position is B.

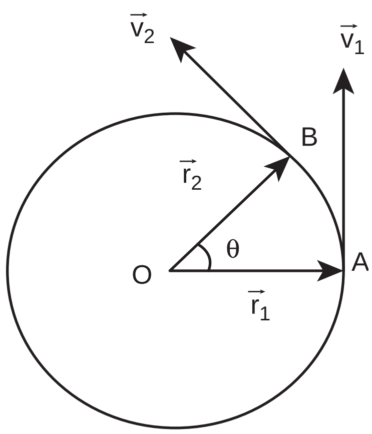


Figure 2.45 Angular displacement

Then,

The angle described by the particle about the axis of rotation (or centre O) in a given time is called angular displacement.

i.e., angular displacement $= \angle AOB = \theta$

The unit of angular displacement is radian.

The angular displacement (θ) in radian is related to arc length S (AB) and radius r as

$$\theta = \frac{S}{r}, \quad \text{or} \quad S = r\theta$$

Angular velocity ($\vec{\omega}$)

The rate of change of angular displacement is called angular velocity.

If θ is the angular displacement in time t, then the angular velocity ω is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The unit of angular velocity is radian per second (rad s^{-1}). The direction of angular velocity is along the axis of rotation following the right hand rule. This is shown in Figure 2.46.

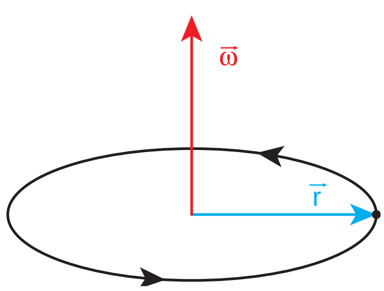


Figure 2.46 Direction of angular velocity

i) Angular acceleration (α)

The rate of change of angular velocity is called angular acceleration.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The angular acceleration is also a vector quantity which need not be in the same direction as angular velocity.

Tangential acceleration

Consider an object moving along a circle of radius r. In a time Δt, the object travels an arc distance Δs as shown in Figure 2.47. The corresponding angle subtended is $\Delta\theta$

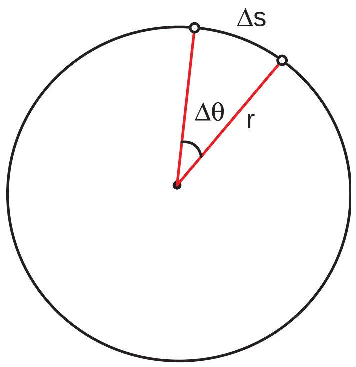


Figure 2.47 Circular motion

The Δs can be written in terms of $\Delta\theta$ as,

$$\Delta s = r\Delta\theta \quad (2.35)$$

In a time Δt, we have

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \quad (2.36)$$

In the limit $\Delta t \to 0$, the above equation becomes

$$\frac{ds}{dt} = r\omega \quad (2.37)$$

Here $\frac{ds}{dt}$ is linear speed (v) which is tangential to the circle and ω is angular speed. So equation (2.37) becomes

$$v = r\omega \quad (2.38)$$

which gives the relation between linear speed and angular speed.

> **Note** The direction of linear velocity $\vec{v}$ is tangential to the circle, whereas the direction of angular velocity $\vec{\omega}$ is along the axis of rotation. The radius is also represented as a vector $\vec{r}$ directed radially from the centre of the circle.

Equation (2.38) is true only for circular motion. In general the relation between linear and angular velocity is given by

$$\vec{v} = \vec{\omega} \times \vec{r} \quad (2.39)$$

For circular motion equation (2.39) reduces to equation (2.38) since $\vec{\omega}$ and $\vec{r}$ are perpendicular to each other.

Differentiating the equation (2.38) with respect to time, we get (since r is constant)

$$\frac{dv}{dt} = \frac{rd\omega}{dt} = r\alpha \quad (2.40)$$

Here $\frac{dv}{dt}$ is the tangential acceleration and is denoted as a_t. $\frac{d\omega}{dt}$ is the angular acceleration α. Then eqn. (2.39) becomes

$$a_t = r\alpha \quad (2.41)$$

The tangential acceleration a_t experienced by an object in circular motion is shown in Figure 2.48.

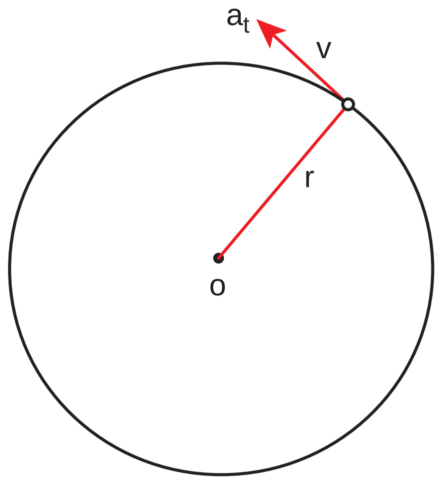


Figure 2.48 Tangential acceleration

Note that the tangential acceleration is in the direction of linear velocity.

2.11.6 Circular Motion

When a point object is moving on a circular path with a constant speed, it covers equal distances on the circumference of the circle in equal intervals of time. Then the object is said to be in uniform circular motion. This is shown in Figure 2.49.

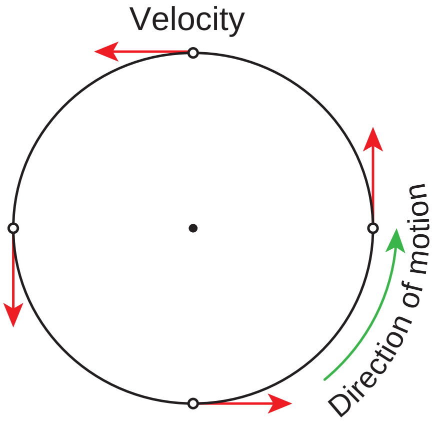


Figure 2.49 Uniform circular motion

In uniform circular motion, the velocity is always changing but speed remains the same. Physically it implies that magnitude of velocity vector remains constant and only the direction changes continuously.

If the velocity changes in both speed and direction during the circular motion, we get non uniform circular motion.

Centripetal acceleration

As seen already, in uniform circular motion the velocity vector turns continuously without changing its magnitude (speed), as shown in Figure 2.50.

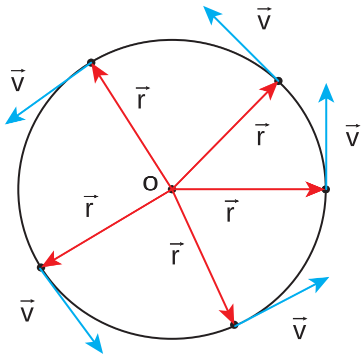


Figure 2.50 Velocity in uniform circular motion

Note that the length of the velocity vector (blue) is not changed during the motion, implying that the speed remains constant. Even though the velocity is tangential at every point in the circle, the acceleration is acting towards the centre of the circle. This is called centripetal acceleration. It always points towards the centre of the circle. This is shown in the Figure 2.51.

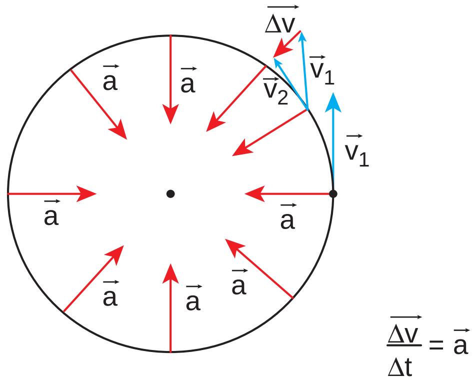


Figure 2.51 Centripetal acceleration

The centripetal acceleration is derived from a simple geometrical relationship between position and velocity vectors (Figure 2.48 or Figure 2.52)

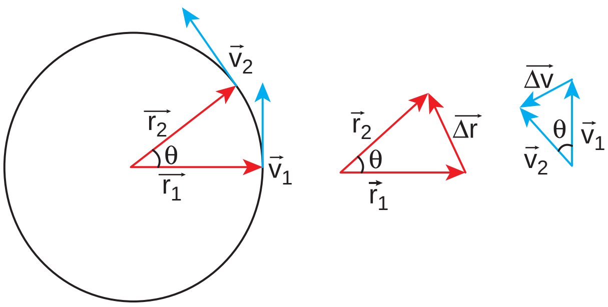


Figure 2.52 Geometrical relationship between the postion and velocity vectors

Let the directions of position and velocity vectors shift through the same angle θ in a small interval of time Δt, as shown in Figure 2.52. For uniform circular motion, $r = |\vec{r}_1| = |\vec{r}_2|$ and $v = |\vec{v}_1| = |\vec{v}_2|$. If the particle moves from position vector $\vec{r}_1$ to $\vec{r}_2$, the displacement is given by $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ and the change in velocity from $\vec{v}_1$ to $\vec{v}_2$ is given by $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$. The magnitudes of the displacement Δr and of Δv satisfy the following relation

$$\frac{\Delta r}{r} = -\frac{\Delta v}{v} = \theta$$

Here the negative sign implies that Δv points radially inward, towards the centre of the circle.

$$\Delta v = -v\left(\frac{\Delta r}{r}\right)$$

$$\text{Then,} \quad a = \frac{\Delta v}{\Delta t} = -\frac{v}{r}\left(\frac{\Delta r}{\Delta t}\right) = -\frac{v^2}{r}$$

For uniform circular motion $v = \omega r$, where ω is the angular velocity of the particle about the centre. Then the centripetal acceleration can be written as

$$a = -\omega^2 r$$

Note The magnitude of centripetal acceleration is constant for uniform circular motion. But centripetal acceleration is not constant, but takes different directions during the motion.

Non uniform circular motion

If the speed of the object in circular motion is not constant, then we have non-uniform circular motion. For example, when the bob attached to a string moves in vertical circle, the speed of the bob is not the same at all time. Whenever the speed is not same in circular motion, the particle will have both centripetal and tangential acceleration as shown in the Figure 2.53.

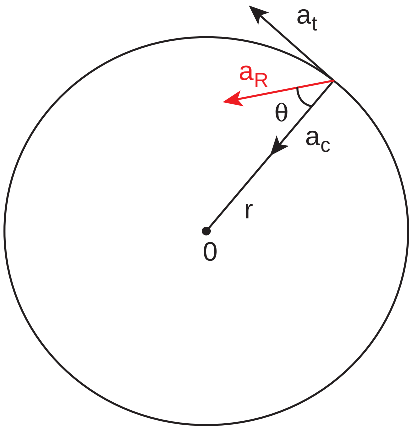


Figure 2.53 Resultant acceleration (a_R) in non uniform circular motion

The resultant acceleration is obtained by vector sum of centripetal and tangential acceleration.

Since centripetal acceleration is $\frac{v^2}{r}$, the magnitude of this resultant acceleration is given by $a_R = \sqrt{a_t^2 + \left(\frac{v^2}{r}\right)^2}$.

This resultant acceleration makes an angle θ with the radius vector as shown in Figure 2.53.

This angle is given by $\tan\theta = \frac{a_t}{\left(v^2/r\right)}$.

EXAMPLE 2.40

A particle moves in a circle of radius 10 m. Its linear speed is given by $v = 3t$ where t is in second and v is in m s^{-1}.

(a) Find the centripetal and tangential acceleration at t = 2 s.

(b) Calculate the angle between the resultant acceleration and the radius vector.

Solution

The linear speed at t = 2 s

$$v = 3t = 6 m\, s^{-1}$$

The centripetal acceleration at t = 2 s is

$$a_c = \frac{v^2}{r} = \frac{(6)^2}{10} = 3.6 m\, s^{-2}$$

The tangential acceleration is $a_t = \frac{dv}{dt} = 3 m\, s^{-2}$

The angle between the radius vector with resultant acceleration is given by

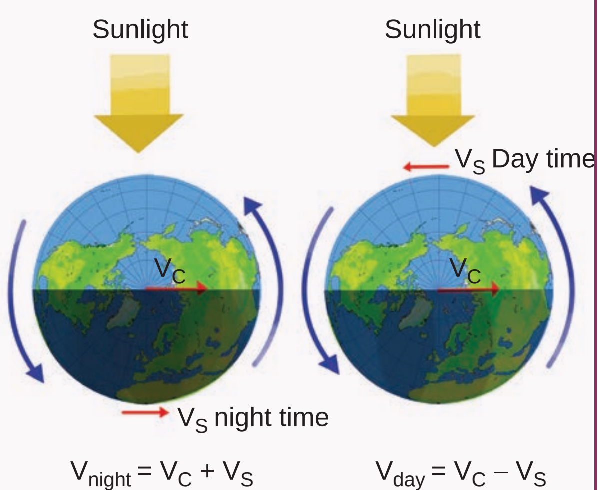
$$\tan\theta = \frac{a_t}{a_c} = \frac{3}{3.6} = 0.833$$

$$\theta = \tan^{-1}(0.833) = 0.69 \text{ radian}$$

In terms of degree $\theta = 0.69 \times 57.27° \approx 40°$

Are we moving with the same speed both day and night with respect to Sun?

Earth orbits the Sun in an elliptical orbit. Let us specify the velocity of the centre of Earth with respect to Sun as $\vec{v}_c$. This $\vec{v}_c$ is due to the elliptical motion of the Earth around the Sun. We know that at the same time Earth is also spinning on its own axis. Due to this spinning, all objects on the surface of the Earth undergo circular motion with velocity $(\vec{v}_s)$ with respect to the axis of rotation of the Earth. At night both $\vec{v}_c$ and $\vec{v}_s$ are either in the same direction or at an acute angle with each other. So, the velocity of an object on the surface of Earth with respect to Sun at night time is $\vec{v}_{night} = \vec{v}_c + \vec{v}_s$. During the day $\vec{v}_c$ and $\vec{v}_s$ are either in opposite directions or at an obtuse angle with each other. So, the velocity of the object with respect to Sun at day time $\vec{v}_{day} = \vec{v}_c - \vec{v}_s$. From this, we can conclude that any object on the surface of the Earth travels faster with respect to Sun during night than during day time. This happens due to the rotation of the Earth.



Kinematic Equations of circular motion

If an object is in circular motion with constant angular acceleration α, we can derive kinematic equations for this motion, analogous to those for linear motion.

Let us consider a particle executing circular motion with initial angular velocity ω_0. After a time interval t it attains a final angular velocity ω. During this time, it covers an angular displacement θ. Because of the change in angular velocity there is an angular acceleration α.

The kinematic equations for circular motion are easily written by following the kinematic equations for linear motion in section 2.4.3

The linear displacement (s) is replaced by the angular displacement (θ).

The velocity (v) is replaced by angular velocity (ω).

The acceleration (a) is replaced by angular acceleration (α).

The initial velocity (u) is replaced by the initial angular velocity (ω_0).

By following this convention, kinematic equations for circular motion are as in the table given below.

Kinematic equations for linear motion	Kinematic equations for angular motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \frac{(v+u)t}{2}$	$\theta = \frac{(\omega_0 + \omega)t}{2}$

Note

Kinematic equations for linear motion are applicable for only constant acceleration. Similarly kinematic equations for angular motion are applicable to only constant angular acceleration.

EXAMPLE 2.41

A particle is in circular motion with an acceleration $\alpha = 0.2\, rad\, s^{-2}$.

(a) What is the angular displacement made by the particle after 5 s?

(b) What is the angular velocity at t = 5 s?. Assume the initial angular velocity is zero.

Solution

Since the initial angular velocity is zero $(\omega_0 = 0)$.

The angular displacement made by the particle is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \frac{1}{2} \times 2 \times 10^{-1} \times 25 = 2.5\, rad$$

In terms of degree

$$\theta = 2.5 \times 57.27° \approx 143°$$

SUMMARY

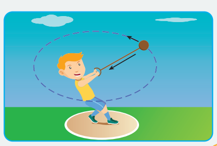
- A state of rest or of motion is defined with respect to a frame of reference.
- In Physics, we conventionally follow a right handed Cartesian coordinate system to explain the motion of objects
- To explain linear motion the concept of point mass is used.
- A vector is a quantity which has both magnitude and direction. A scalar has only magnitude.
- The length of a vector is called magnitude or norm of the vector
- In a Cartesian coordinate system the unit vectors are orthogonal to each other.
- Vectors can be added using either the triangular law of addition or the parallelogram law of addition.
- Any vector can be resolved into three components with respect to a Cartesian coordinate system
- The magnitude or norm of a vector $\vec{A}$ is given by $\left|\vec{A}\right| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- If two vectors are equal, then their corresponding individual components should be separately equal.
- The position vector of a particle with respect to a Cartesian coordinate system is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- The scalar product of two vectors is defined by $\vec{A}.\vec{B} = AB\cos\theta$ (θ *is the angle between* $\vec{A}$ *and* $\vec{B}$)
- The vector product of two vectors is defined by $\vec{A}\times\vec{B} = \left(AB\sin\theta\right)\hat{n}$. The direction of $\hat{n}$ can be found using right hand thumb rule or right hand cork screw rule.
- In physics, scalar and vector products are used to describe various concepts.
- Distance is the total path length travelled by the particle and displacement is the difference between final and initial positions. Distance is a scalar quantity and displacement is a vector
- Average velocity is defined as $\vec{v}_{avg} = \dfrac{\Delta\vec{r}}{\Delta t}$ and instantaneous velocity is defined as $\vec{v} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$. Both average velocity and instantaneous velocity are vector quantities.
- Momentum is defined as $\vec{p} = m\vec{v}$.

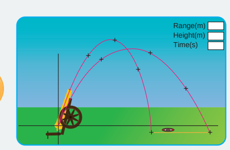
SUMMARY (*cont*)

- The average acceleration is defined as $\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$ and instantaneous acceleration is defined as $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$
- For constant acceleration, kinematic equations can be used to analyse motion of a particle.
- In projectile motion for which acceleration is uniform, the path traced by the particle is a parabola.
- The maximum height and range of the particle in projectile motion depend inversely on acceleration due to gravity g.
- The angular displacement of the particle is defined by $\theta = \frac{s}{r}$ and angular velocity $\vec{\omega} = \frac{d\vec{\theta}}{dt}$
- The relation between the linear velocity and angular velocity is given by $\vec{v} = \vec{\omega} \times \vec{r}$
- The centripetal acceleration is given by $a_c = -\frac{v^2}{r}$ or $-\omega^2 r$ and is always directed towards the centre of the circle.

CONCEPT MAP

- Measuring Motion
  - Frame of Reference
    - Vector / Scalars
    - Position Vector Distance Displacement Vector
      - Calculus
      - Velocity Average Velocity Speed, Average Speed
        - Acceleration
          - Average Acceleration
          - Circular Motion
            - Uniform circular motion
            - Non-Uniform Circular motion
              - Kinematic Equation
          - Motion at constant acceleration
            - Kinematic Equation
              - Projectile Motion

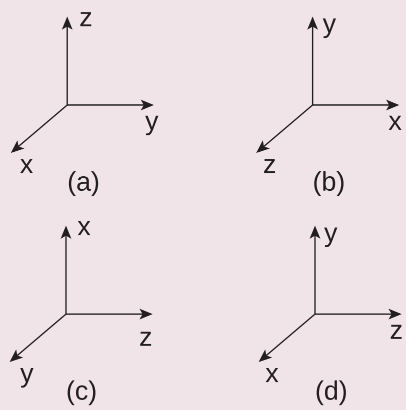




EVALUATION

I. Multi Choice Question

1. Which one of the following Cartesian coordinate systems is not followed in physics?



2. Identify the unit vector in the following.

(a) $\hat{i}+\hat{j}$ (b) $\dfrac{\hat{i}}{\sqrt{2}}$

(c) $\hat{k}-\dfrac{\hat{j}}{\sqrt{2}}$ (d) $\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$

3. Which one of the following physical quantities cannot be represented by a scalar?

(a) Mass

(b) length

(c) momentum

(d) magnitude of acceleration

4. Two objects of masses m_1 and m_2 fall from the heights h_1 and h_2 respectively. The ratio of the magnitude of their momenta when they hit the ground is (AIPMT 2012)

(a) $\sqrt{\dfrac{h_1}{h_2}}$ (b) $\sqrt{\dfrac{m_1h_1}{m_2h_2}}$

(c) $\dfrac{m_1}{m_2}\sqrt{\dfrac{h_1}{h_2}}$ (d) $\dfrac{m_1}{m_2}$

5. If a particle has negative velocity and negative acceleration, its speed

(a) increases (b) decreases

(c) remains same (d) zero

6. If the velocity is $\vec{v}=2\hat{i}+t^2\hat{j}-9\vec{k}$, then the magnitude of acceleration at t = 0.5 s is

(a) 1m s^{-2} (b) 2 m s^{-2}

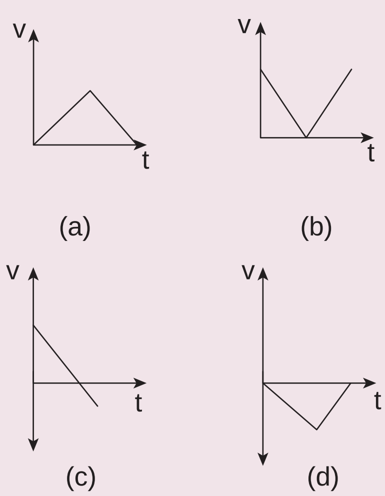
(c) zero (d) -1 m s^{-2}

7. If an object is dropped from the top of a building and it reaches the ground at t = 4 s, then the height of the building is (ignoring air resistance) (g = 9.8 ms^{-2})

(a) 77.3 m (b) 78.4 m

(c) 80.5 m (d) 79.2 m

8. A ball is projected vertically upwards with a velocity v. It comes back to ground in time t. Which v-t graph shows the motion correctly? (NSEP 00-01)

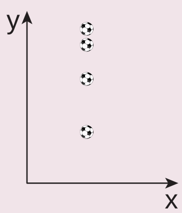


9. If one object is dropped vertically downward and another object is thrown horizontally from the same height, then the ratio of vertical distance covered by both objects at any instant t is

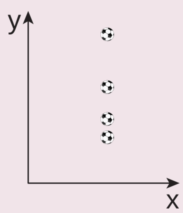
(a) 1 (b) 2

(c) 4 (d) 0.5

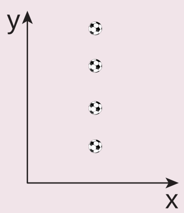
10. A ball is dropped from some height towards the ground. Which one of the following represents the correct motion of the ball?



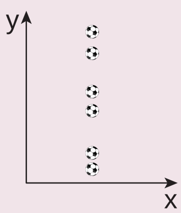
(a)



(b)



(c)



(d)

11. If a particle executes uniform circular motion in the xy plane in clock wise direction, then the angular velocity is in

(a) +y direction

(b) +z direction

(c) -z direction

(d) -x direction

12. If a particle executes uniform circular motion, choose the correct statement (NEET 2016)

(a) The velocity and speed are constant.

(b) The acceleration and speed are constant.

(c) The velocity and acceleration are constant.

(d) The speed and magnitude of acceleration are constant.

13. If an object is thrown vertically up with the initial speed u from the ground, then the time taken by the object to return back to ground is

(a) $\frac{u^2}{2g}$ (b) $\frac{u^2}{g}$

(c) $\frac{u}{2g}$ (d) $\frac{2u}{g}$

14. Two objects are projected at angles 30° and 60° respectively with respect to the horizontal direction. The range of two objects are denoted as R_{30^0} and R_{60^0}. Choose the correct relation from the following

(a) $R_{30^0} = R_{60^0}$

(b) $R_{30^0} = 4R_{60^0}$

(c) $R_{30^0} = \frac{R_{60^0}}{2}$

(d) $R_{30^0} = 2\,R_{60^0}$

15. An object is dropped in an unknown planet from height 50 m, it reaches the ground in 2 s. The acceleration due to gravity in this unknown planet is

(a) g = 20 m s^{-2} (b) g = 25 m s^{-2}

(c) g = 15 m s^{-2} (d) g = 30 m s^{-2}

Answers

1) d	**2)** d	**3)** c	**4)** c	**5)** a
6) a	**7)** b	**8)** c	**9)** a	**10)** a
11) c	**12)** d	**13)** d	**14)** a	**15)** b

II. Short Answer Questions

1. Explain what is meant by Cartesian coordinate system?

2. Define a vector. Give examples

3. Define a scalar. Give examples

4. Write a short note on the scalar product between two vectors.

5. Write a short note on vector product between two vectors.
6. How do you deduce that two vectors are perpendicular?
7. Define displacement and distance.
8. Define velocity and speed.
9. Define acceleration.
10. What is the difference between velocity and average velocity.
11. Define a radian.
12. Define angular displacement and angular velocity.
13. What is non uniform circular motion?
14. Write down the kinematic equations for angular motion.
15. Write down the expression for angle made by resultant acceleration and radius vector in the non uniform circular motion.

III. Long Answer Questions

1. Explain in detail the triangle law of addition.
2. Discuss the properties of scalar and vector products.
3. Derive the kinematic equations of motion for constant acceleration.
4. Derive the equations of motion for a particle (a) falling vertically (b) projected vertically
5. Derive the equation of motion, range and maximum height reached by the particle thrown at an oblique angle θ with respect to the horizontal direction.
6. Derive the expression for centripetal acceleration.
7. Derive the expression for total acceleration in the non uniform circular motion.

IV. Exercises

1. The position vectors particle has length 1m and makes 30° with the x-axis. What are the lengths of the x and y components of the position vector?

[Ans: $l_x = \frac{\sqrt{3}}{2}, l_y = 0.5$]

2. A particle has its position moved from $\vec{r}_1 = 3\hat{i} + 4\hat{j}$ to $\vec{r}_2 = \hat{i} + 2\hat{j}$. Calculate the displacement vector $(\Delta\vec{r})$ and draw the $\vec{r}_1$, $\vec{r}_2$ and $\Delta\vec{r}$ vector in a two dimensional Cartesian coordinate system.

[Ans: $\Delta\vec{r} = -2\hat{i} - 2\hat{j}$]

3. Calculate the average velocity of the particle whose position vector changes from $\vec{r}_1 = 5\hat{i} + 6\hat{j}$ to $\vec{r}_2 = 2\hat{i} + 3\hat{j}$ in a time 5 second.

[Ans: $\vec{v}_{avg} = -\frac{3}{5}\left(\hat{i} + \hat{j}\right)$]

4. Convert the vector $\vec{r} = 3\hat{i} + 2\hat{j}$ into a unit vector.

[Ans: $\hat{r} = \frac{\left(3\hat{i} + 2\hat{j}\right)}{\sqrt{13}}$]

5. What are the resultants of the vector product of two given vectors given by

$$\vec{A} = 4\hat{i} - 2\hat{j} + \hat{k} \text{ and } \vec{B} = 5\hat{i} + 3\hat{j} - 4\hat{k}?$$

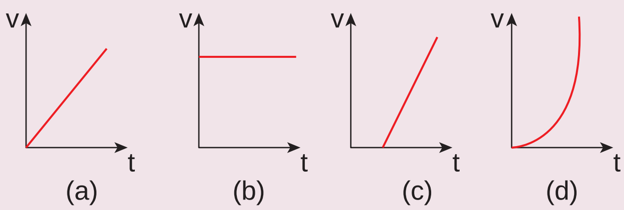
[Ans: $5\hat{i} + 21\hat{j} + 22\hat{k}$]

6. An object at an angle such that the horizontal range is 4 times of the

maximum height. What is the angle of projection of the object?

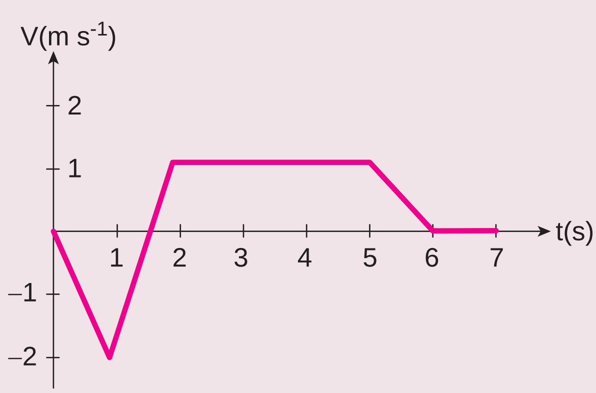
[Ans: $\theta = 45°$]

7. The following graphs represent velocity – time graph. Identify what kind of motion a particle undergoes in each graph.



[Ans: (a) $\vec{a} = constant$ (b) $\vec{v} = constant$ (c) $\vec{a} = constant$ but greater than first graph (d) $\vec{a}$ *is variable*]

8. The following velocity–time graph represents a particle moving in the positive x–direction. Analyse its motion from 0 to 7 s. Calculate the displacement covered and distance travelled by the particle from 0 to 2 s.



[Ans: distance=1.75 m, displacement = -1.25 m]

9. A particle is projected at an angle of θ with respect to the horizontal direction. Match the following for the above motion.

(a) v_x – decreases and increases

(b) v_y – remains constant

(c) Acceleration – varies

(d) Position vector – remains downward

[Ans: $v_x = remains\ constant$,
$v_y = decreases\ and\ increases$,
$a = remains\ downward$,
$r = varies$]

10. A water fountain on the ground sprinkles water all around it. If the speed of the water coming out of the fountain is v. Calculate the total area around the fountain that gets wet.

[Ans: Area = $\frac{\pi v^4}{g^2}$]

11. The following table gives the range of a particle when thrown on different planets. All the particles are thrown at the same angle with the horizontal and with the same initial speed. Arrange the planets in ascending order according to their acceleration due to gravity, (g value).

Planet	Range
Jupiter	50 m
Earth	75 m
Mars	90 m
Mercury	95 m

[Ans: $g_{jupiter}$ is greater, $g_{mercury}$ is smaller]

12. The resultant of two vectors A and B is perpendicular to vector A and its magnitude is equal to half of the magnitude of vector B. Then the angle between A and B is

a) 30° b) 45°

c) 150° d) 120°

[Ans: $\theta = 150°$]

13. Compare the components for the following vector equations

a) $T\hat{j} - mg\hat{j} = ma\hat{j}$ b) $\vec{T} + \vec{F} = \vec{A} + \vec{B}$

c) $\vec{T} - \vec{F} = \vec{A} - \vec{B}$ d) $T\hat{j} + mg\hat{j} = ma\hat{j}$

[Ans: (a) $T - mg = ma$
(b) $T_x + F_x = A_x + B_x$ etc.]

14. Calculate the area of the triangle for which two of its sides are given by the vectors $\vec{A} = 5\hat{i} - 3\hat{j}, \vec{B} = 4\hat{i} + 6\hat{j}$

[Ans: Area = 21 squared units]

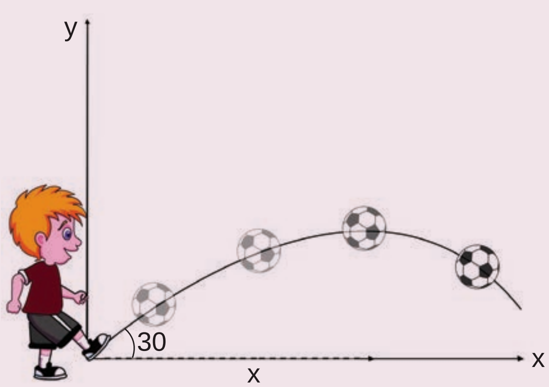
15. If Earth completes one revolution in 24 hours, what is the angular displacement made by Earth in one hour. Express your answer in both radian and degree.

[Ans: $\theta = 15° \, or \, \frac{\pi}{12}$]

16. A object is thrown with initial speed 5 m s^{-1} with an angle of projection 30°. What is the height and range reached by the particle?

[Ans: height = 0.318 m Range =2.21 m]

17. A foot-ball player hits the ball with speed 20 m s^{-1} with angle 30° with respect to horizontal direction as shown in the figure. The goal post is at distance of 40 m from him. Find out whether ball reaches the goal post?



[Ans: Ball will not reach the goal post. The range =35.3 m]

18. If an object is thrown horizontally with an initial speed 10 m s^{-1} from the top of a building of height 100 m. what is the horizontal distance covered by the particle?

[Ans: R= 45 m]

19. An object is executing uniform circular motion with an angular speed of $\frac{\pi}{12}$ radian per second. At t = 0 the object starts at an angle θ = 0 What is the angular displacement of the particle after 4 s?

[Ans: 60°]

20. Consider the x–axis as representing east, the y–axis as north and z–axis as vertically upwards. Give the vector representing each of the following points.

a) 5 m north east and 2 m up

b) 4 m south east and 3 m up

c) 2 m north west and 4 m up

[Ans: (a) $\frac{5(\hat{i}+\hat{j})}{\sqrt{2}} + 2\hat{k}$ (b) $\frac{4(\hat{i}-\hat{j})}{\sqrt{2}} + 3\hat{k}$
(c) $(-\hat{i} + \hat{j})\sqrt{2} + 4\hat{k}$]

21. The Moon is orbiting the Earth approximately once in 27 days, what is the angle transversed by the Moon per day?

[Ans: 13°3']

22. An object of mass m has angular acceleration $\alpha = 0.2 \, rad \, s^{-2}$. What is the angular displacement covered by the object after 3 second? (Assume that the object started with angle zero with zero angular velocity).

[Ans: 0.9 rad or 51°]

BOOKS FOR REFERENCE

1. Charles Kittel, Walter Knight, Malvin Ruderman, Carl Helmholtz and Moyer, *Mechanics,* 2nd edition, Mc Graw Hill Pvt Ltd,
2. A.P.French, *Newtonian Mechanics,* Viva-Norton Student edition
3. SomnathDatta, *Mechanics,* Pearson Publication
4. H.C.Verma, *Concepts of physics* volume 1 and Volume 2, Bharati Bhawan Publishers
5. Serway and Jewett, *Physics for scientist and Engineers with modern physics*, Brook/Coole publishers, Eighth edition
6. Halliday, Resnick & Walker, *Fundamentals of Physics*, Wiley Publishers, 10th edition

ICT CORNER

Projectile motion

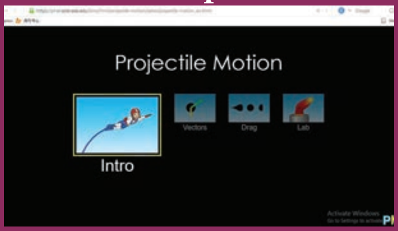
HIT THE TARGET

Through this activity you will be able to understand the velocity variation and Different angles of projection and range

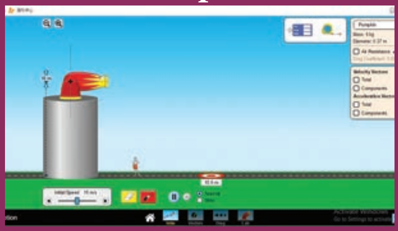
STEPS:

- Type the given URL (or) Scan the QR Code. You can see "Projectile Motion" PhET simulation page. Click 'Intro' to initiate the activity.
- Click the red coloured shoot button. Blast a ball out of a cannon, and challenge yourself to hit the target.
- Drag 'up & down' button to change the height of the cylinder. Click left and right button to change the speed of the cannon ball.
- Drag the target box and fix the target to measure time, range and height. Drag the Meter tape to measure the length from cannon. On the right side top, mark in the corresponding boxes to know the velocity vectors and acceleration vectors.

Step1



Step2



Step3

Step4

Timeline Project's URL:

https://phet.colorado.edu/sims/html/projectile-motion/latest/projectile-motion_en.html

* Pictures are indicative only.

* If browser requires, allow **Flash Player** or **Java Script** to load the page.

UNIT 3

LAWS OF MOTION

"In the beginning there was a mechanics" – Von Laue

Learning Objectives

In this unit, the student is exposed to

- Newton's laws
- logical connection between laws of Newton
- free body diagram and related problems
- law of conservation of momentum
- role of frictional forces
- centripetal and centrifugal forces
- origin of centrifugal force

AWX3Z

3.1 INTRODUCTION

Each and every object in the universe interacts with every other object. The cool breeze interacts with the tree. The tree interacts with the Earth. In fact, all species interact with nature. But, what is the difference between a human's interaction with nature and that of an animal's. Human's interaction has one extra quality. We not only interact with nature but also try to understand and explain natural phenomena scientifically.

In the history of mankind, the most curiosity driven scientific question asked was about motion of objects–'How things move?' and 'Why things move?' Surprisingly, these simple questions have paved the way for development from early civilization to the modern technological era of the 21st century.

Objects move because something pushes or pulls them. For example, if a book is at rest, it will not move unless a force is applied on it. In other words, to move an object a force must be applied on it. About 2500 years ago, the famous philosopher, Aristotle, said that '*Force causes motion*'. This statement is based on common sense. But any scientific answer cannot be based on common sense. It must be endorsed with quantitative experimental proof.

In the 15th century, Galileo challenged Aristotle's idea by doing a series of experiments. He said force is not required to maintain motion.

Galileo demonstrated his own idea using the following simple experiment. When a ball rolls from the top of an inclined plane to its bottom, after reaching the ground it moves some distance and continues to move on to another inclined plane of same angle of inclination as shown in the Figure 3.1(a). By increasing the smoothness of both the inclined planes, the ball reach almost the same height(h) from where it was released (L1) in the second plane (L2) (Figure 3.1(b)). The motion of the ball is then observed by varying the angle of inclination of the second plane keeping the same smoothness. If the angle of inclination is reduced, the ball travels longer distance in the second plane to reach the same height (Figure 3.1 (c)). When the angle of inclination is made zero, the ball moves forever in the horizontal direction (Figure 3.1(d)). If the Aristotelian idea were true, the ball would not have moved in the second plane even if its smoothness is made maximum since no

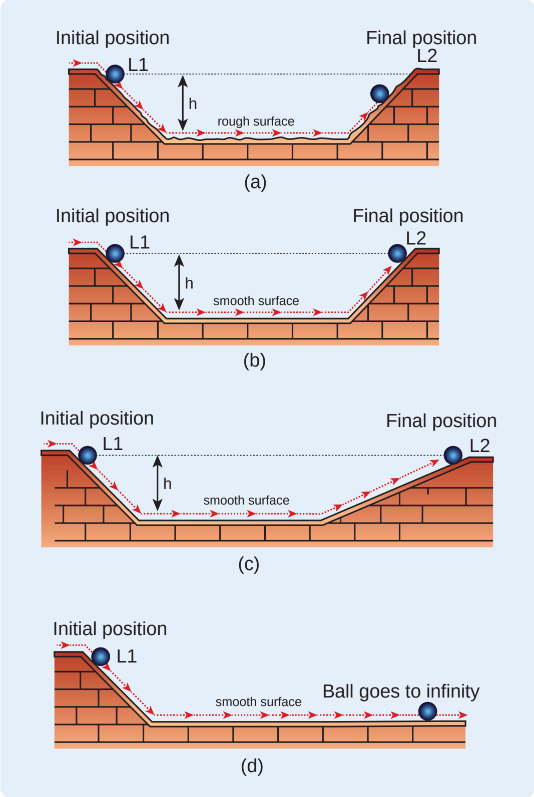


Figure 3.1 Galileo's experiment with the second plane (a) at same inclination angle as the first (b) with increased smoothness (c) with reduced angle of inclination (d) with zero angle of inclination

force acted on it in the horizontal direction. From this simple experiment, Galileo proved that force is not required to maintain motion. An object can be in motion even without a force acting on it.

In essence, Aristotle coupled the motion with force while Galileo decoupled the motion and force.

3.2 NEWTON'S LAWS

Newton analysed the views of Galileo, and other scientist like Kepler and Copernicus on motion and provided much deeper insights in the form of three laws.

3.2.1 Newton's First Law

Every object continues to be in the state of rest or of uniform motion (constant velocity) unless there is external force acting on it.

This inability of objects to move on its own or change its state of motion is called inertia. Inertia means resistance to change its state. Depending on the circumstances, there can be three types of inertia.

1. **Inertia of rest:** When a stationary bus starts to move, the passengers experience a sudden backward push. Due to inertia, the body (of a passenger) will try to

Figure 3.2 Passengers experience a backward push due to inertia of rest

continue in the state of rest, while the bus moves forward. This appears as a backward push as shown in Figure 3.2. *The inability of an object to change its state of rest is called inertia of rest.*

2. **Inertia of motion:** When the bus is in motion, and if the brake is applied suddenly, passengers move forward and hit against the front seat. In this case, the bus comes to a stop, while the body (of a passenger) continues to move forward due to the property of inertia as shown in Figure 3.3. *The inability of an object to change its state of uniform speed (constant speed) on its own is called inertia of motion.*
3. **Inertia of direction:** When a stone attached to a string is in whirling motion, and if the string is cut suddenly, the stone will not continue to move in circular motion but moves tangential to the circle as illustrated in Figure 3.4. This is because the body cannot change its direction of motion without any force acting on it. *The inability of an object to change its direction of motion on its own is called inertia of direction.*

Figure 3.3 Passengers experience a forward push due to inertia of motion

When we say that an object is at rest or in motion with constant velocity, it has a meaning only if it is specified with respect to some reference frames. In physics, any motion has to be stated with respect to a reference frame. It is to be noted that Newton's first law is valid only in certain special reference frames called inertial frames. In fact, Newton's first law defines an inertial frame.

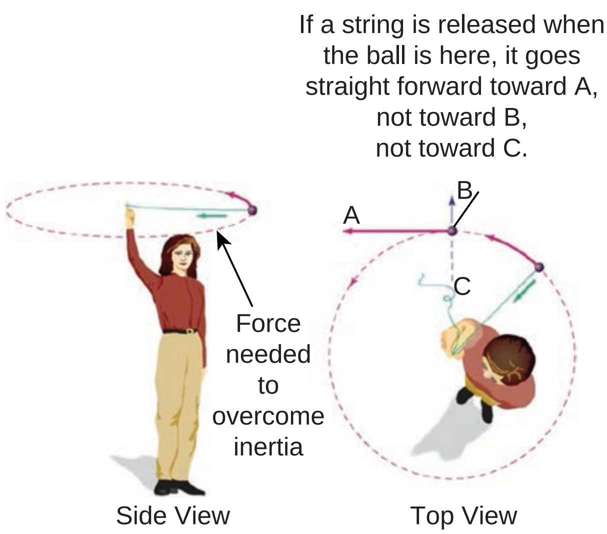


Figure 3.4 A stone moves tangential to circle due to inertia of direction

Inertial Frames

If an object is free from all forces, then it moves with constant velocity or remains at rest when seen from inertial frames. Thus, there exists some special set of frames in which, if an object experiences no force, it moves with constant velocity or remains at rest. But how do we know whether an object is experiencing a force or not? All the objects in the Earth experience Earth's gravitational force. In the ideal case, if an object is in deep space (very far away from any other object), then Newton's first law will be certainly valid. Such deep space can be treated as an inertial frame. But practically it is not possible to reach such deep space and verify Newton's first law.

For all practical purposes, we can treat Earth as an inertial frame because an object on the table in the laboratory appears to be at rest always. This object never picks up acceleration in the horizontal direction since no force acts on it in the horizontal direction. So the laboratory can be taken as an inertial frame for all physics experiments and calculations. For making these conclusions, we analyse only the horizontal motion of the object as there is no horizontal force that acts on it. We should not analyse the motion in vertical direction as the two forces (gravitational force in the downward direction and normal force in upward direction) that act on it makes the net force is zero in vertical direction. Newton's first law deals with the motion of objects in the absence of any force and not the motion under zero net force. Suppose a train is moving with constant velocity with respect to an inertial frame, then an object at rest in the inertial frame (outside the train) appears to move with constant velocity with respect to the train (viewed from within the train). So the train can be treated as an inertial frame. All inertial frames are moving with constant velocity relative to each other. If an object appears to be at rest in one inertial frame, it may appear to move with constant velocity with respect to another inertial frame. For example, in Figure 3.5, the car is moving with uniform velocity v with respect to a person standing (at rest) on the ground. As the car is moving with constant velocity with respect to the person at rest on the ground, both frames (with respect to the car and to the ground) are inertial frames.

Figure 3.5 The person and vehicle are inertial frames

Suppose an object remains at rest on a smooth table kept inside the train, and if the train suddenly accelerates (which we may not sense), the object appears to accelerate backwards even without any force acting on it. It is a clear violation of Newton's first law as the object gets accelerated without being acted upon by a force. It implies that the train is not an inertial frame when it is accelerated. For example, Figure 3.6 shows that car 2 is a non-inertial frame since it moves with acceleration $\vec{a}$ with respect to the ground.

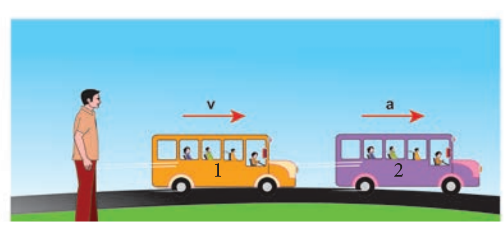


Figure 3.6 Car 2 is a non-inertial frame

These kinds of accelerated frames are called non-inertial frames. A rotating frame is also a non inertial frame since rotation requires acceleration. In this sense, Earth is not really an inertial frame since it has self-rotation and orbital motion. But these rotational effects of Earth can be ignored for the motion involved in our day-to-day life. For example, when an object is thrown, or the time period of a simple pendulum is measured in the physics laboratory, the Earth's self-rotation has very negligible effect on it. In this sense, Earth can be treated as an inertial frame. But at the same time, to analyse the motion of satellites and wind patterns around the Earth, we cannot treat Earth as an inertial frame since its self-rotation has a strong influence on wind patterns and satellite motion.

3.2.2 Newton's Second Law

This law states that

The force acting on an object is equal to the rate of change of its momentum

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{3.1}$$

In simple words, whenever the momentum of the body changes, there must be a force acting on it. The momentum of the object is defined as $\vec{p} = m\vec{v}$. In most cases, the mass of the object remains constant during the motion. In such cases, the above equation gets modified into a simpler form

$$\vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

$$\vec{F} = m\vec{a}. \tag{3.2}$$

The above equation conveys the fact that if there is an acceleration $\vec{a}$ on the body, then there must be a force acting on it. This implies that if there is a change in velocity, then there must be a force acting on the body. The force and acceleration are always in the same direction. Newton's second law was a paradigm shift from Aristotle's idea of motion. According to Newton, the force need not cause the motion but only a change in motion. It is to be noted that *Newton's second law is valid only in inertial frames.* In non-inertial frames Newton's second law cannot be used in this form. It requires some modification.

In the SI system of units, the unit of force is measured in newtons and it is denoted by symbol 'N'.

One Newton is defined as the force which acts on 1 kg of mass to give an acceleration 1 m s^{-2} in the direction of the force.

Aristotle vs. Newton's approach on sliding object

Newton's second law gives the correct explanation for the experiment on the inclined plane that was discussed in section 3.1. In normal cases, where friction is not negligible, once the object reaches the bottom of the inclined plane (Figure 3.1), it travels some distance and stops. Note that it stops because there is a frictional force acting in the direction opposite to its velocity. It is this frictional force that reduces the velocity of the object to zero and brings it to rest. As per Aristotle's idea, as soon as the body reaches the bottom of the plane, it can travel only a small distance and stops because there is no force acting on the object. Essentially, he did not consider the frictional force acting on the object.

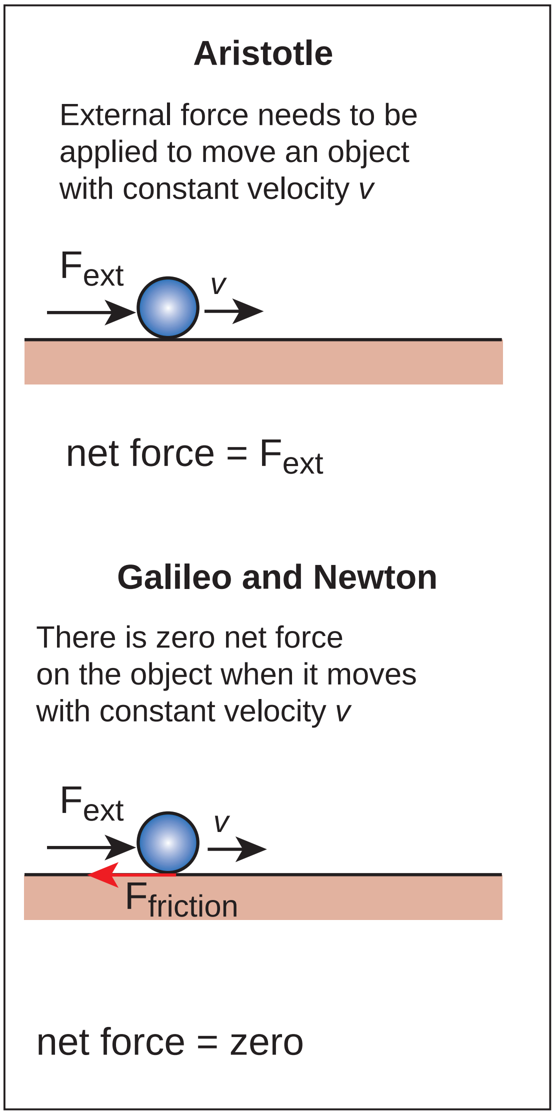


Figure 3.7 Aristotle, Galileo and Newton's approach

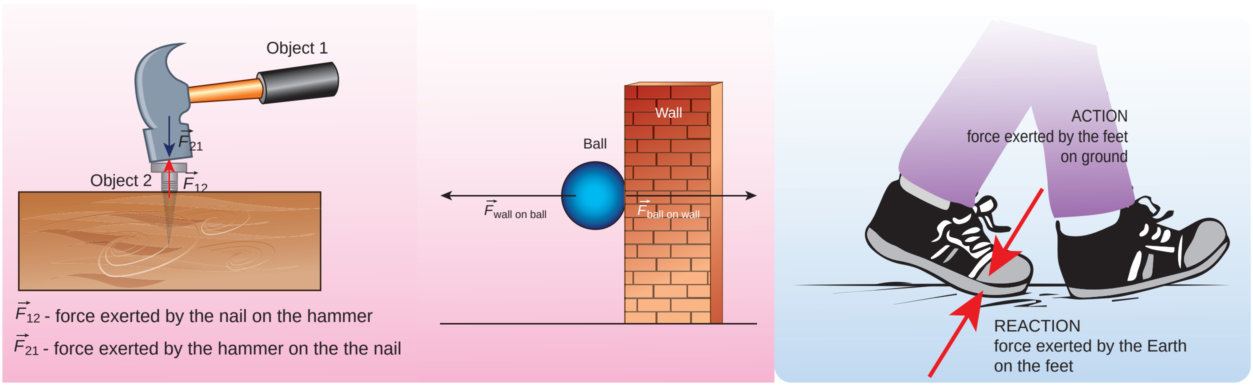
3.2.3 Newton's Third Law

Consider Figure 3.8(a) whenever an object 1 exerts a force on the object 2 ($\vec{F}_{21}$), then object 2 must also exert equal and opposite force on the object 1 ($\vec{F}_{12}$). These forces must lie along the line joining the two objects.

$$\vec{F}_{12} = -\vec{F}_{21}$$

Newton's third law assures that the forces occur as equal and opposite pairs. An isolated force or a single force cannot exist in nature. *Newton's third law states that for every action there is an equal and opposite reaction*. Here, action and reaction pair of forces do not act on the same body but on two different bodies. Any one of the forces can be called as an action force and the other the reaction force. Newton's third law is valid in both inertial and non-inertial frames.

These action-reaction forces are not cause and effect forces. It means that when the object 1 exerts force on the object 2, the object 2 exerts equal and opposite force on the body 1 at the same instant.



(a) (b) (c)

Figure 3.8 Demonstration of Newton's third law (a) Hammer and the nail (b) Ball bouncing off the wall (c) Walking on the floor with friction

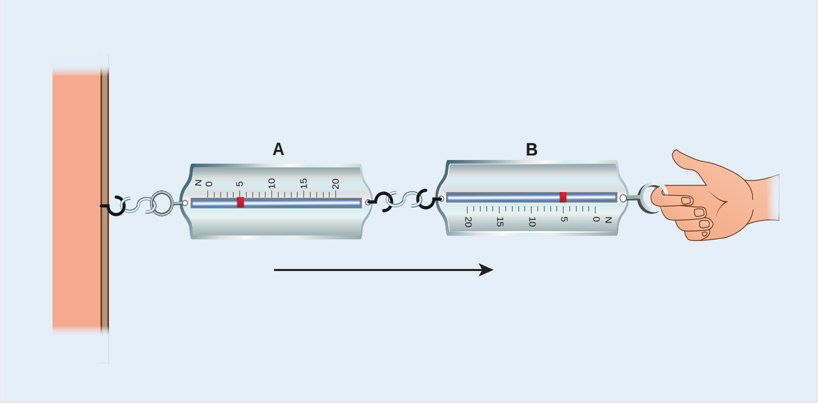
ACTIVITY

Verification of Newton's third law

Attach two spring balances as shown in the figure. Fix one end with rigid support and leave the other end free, which can be pulled with the hand.

Pull one end with some force and note the reading on both the balances.

Repeat the exercise a number of times.



Note The reading in the spring balance A is due to the force given by spring balance B. The reading in the spring balance B is due to the reaction force given by spring balance A. Note that according to Newton's third law, both readings (force) are equal.

1JBKT

3.2.4 Discussion on Newton's Laws

1. Newton's laws are vector laws. The equation $\vec{F} = m\vec{a}$ is a vector equation and essentially it is equivalent to three scalar equations. In Cartesian coordinates, this equation can be written as $F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = ma_x\hat{i} + ma_y\hat{j} + ma_z\hat{k}$.
By comparing both sides, the three scalar equations are

$F_x = ma_x$ The acceleration along the x direction depends only on the component of force acting along the x-direction.

$F_y = ma_y$ The acceleration along the y direction depends only on the component of force acting along the y-direction.

$F_z = ma_z$ The acceleration along the z direction depends only on the component of force acting along the z-direction.

From the above equations, we can infer that the force acting along y direction cannot alter the acceleration along x direction. In the same way, F_z cannot affect a_y and a_x. This understanding is essential for solving problems.

2. The acceleration experienced by the body at time t depends on the force which acts on the body at that instant of time. It does not depend on the force which acted on the body before the time t. This can be expressed as

$$\vec{F}(t) = m\vec{a}(t)$$

Acceleration of the object does not depend on the previous history of the force. For example, when a spin bowler or a fast bowler throws the ball to the batsman, once the ball leaves the hand of the bowler, it experiences only gravitational force and air frictional force. The acceleration of the ball is independent of how the ball was bowled (with a lower or a higher speed).

3. In general, the direction of a force may be different from the direction of motion. Though in some cases, the object may move in the same direction as the direction of the force, it is not always true. A few examples are given below.

Case 1: Force and motion in the same direction

When an apple falls towards the Earth, the direction of motion (direction of velocity) of the apple and that of force are in the same downward direction as shown in the Figure 3.9 (a).

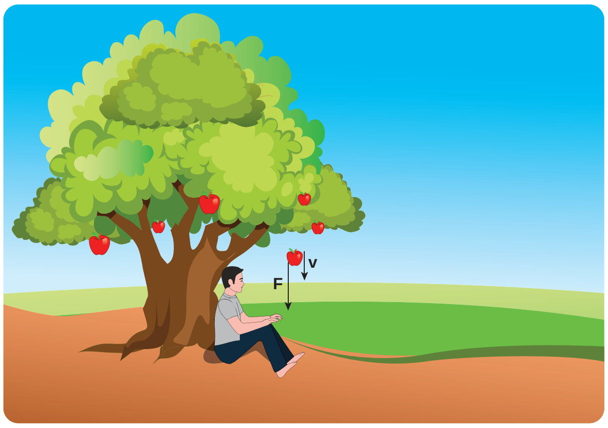


Figure 3.9 (a) Force and motion in the same direction

Case 2: Force and motion not in the same direction

The Moon experiences a force towards the Earth. But it actually moves in elliptical orbit. In this case, the direction of the force is different from the direction of motion as shown in Figure 3.9 (b).

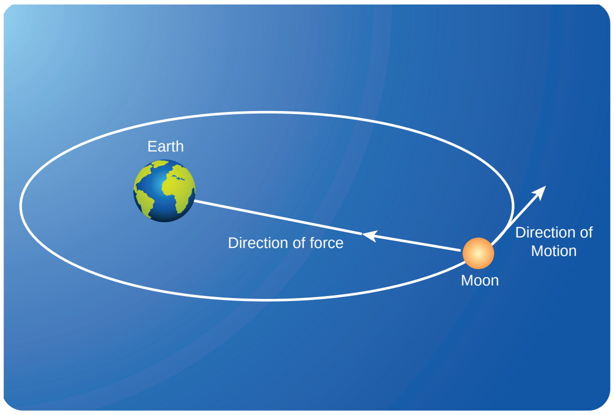


Figure 3.9 (b) Moon orbiting in elliptical orbit around the Earth

Case 3: Force and motion in opposite direction

If an object is thrown vertically upward, the direction of motion is upward, but gravitational force is downward as shown in the Figure 3.9 (c).

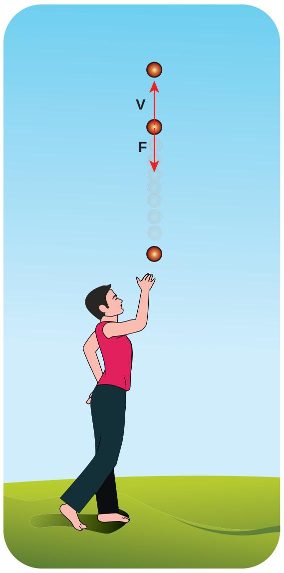


Figure 3.9 (c) Force and direction of motion are in opposite directions

Case 4: Zero net force, but there is motion

When a raindrop gets detached from the cloud it experiences both downward gravitational force and upward air drag force. As it descends towards the Earth, the upward air drag force increases and after a certain time, the upward air drag force cancels the downward gravity. From then on the raindrop moves at constant velocity till it touches the surface of the Earth. Hence the raindrop comes with zero net force, therefore with zero acceleration but with non-zero terminal velocity. It is shown in the Figure 3.9 (d).

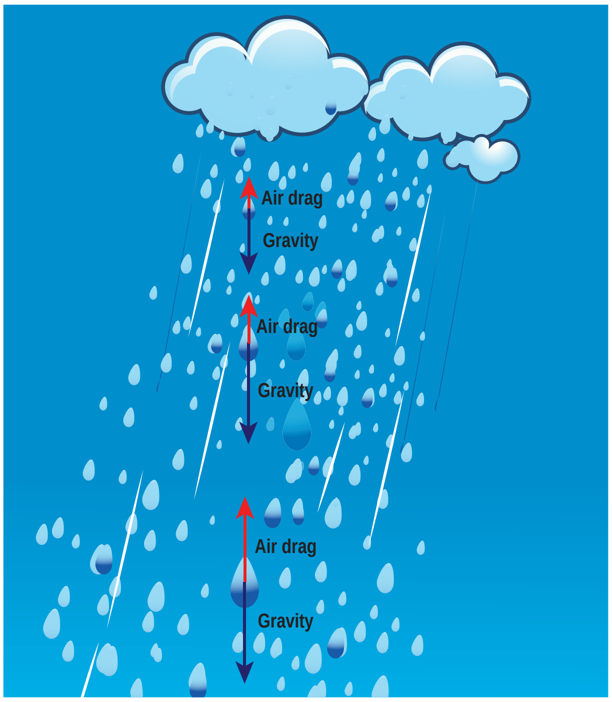


Figure 3.9 (d) Zero net force and non zero terminal velocity

4. If multiple forces $\vec{F}_1, \vec{F}_2, \vec{F}_3 \ldots \vec{F}_n$ act on the same body, then the total force $(\vec{F}_{net})$ is equivalent to the vectorial sum of the individual forces. Their net force provides the acceleration.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$$

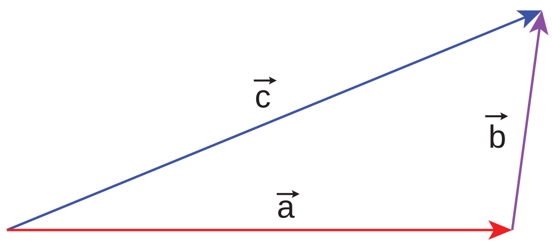


Figure 3.10 Vector addition of forces

Newton's second law for this case is

$$\vec{F}_{net} = m\vec{a}$$

In this case the direction of acceleration is in the direction of net force.

Example

Bow and arrow

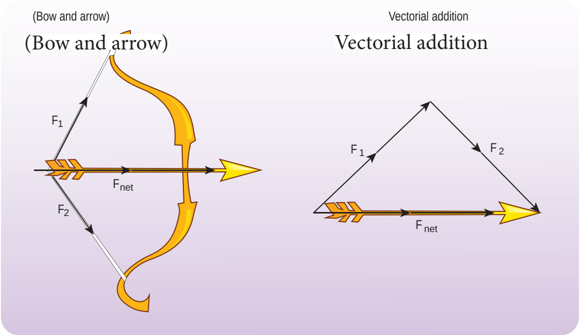


Figure 3.11 Bow and arrow – Net force is on the arrow

5. Newton's second law can also be written in the following form.

Since the acceleration is the second derivative of position vector of the body $\left(\vec{a} = \frac{d^2\vec{r}}{dt^2}\right)$, the force on the body is

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}.$$

From this expression, we can infer that Newton's second law is basically a second order ordinary differential equation and whenever the second derivative of position vector is not zero, there must be a force acting on the body.

6. If no force acts on the body then Newton's second law, $m\frac{d\vec{v}}{dt} = 0$.

   It implies that $\vec{v} = constant$. It is essentially Newton's first law. It implies that the second law is consistent with the first law. However, it should not be thought of as the reduction of second law to the first when no force acts on the object. Newton's first and second laws are independent laws. They can internally be consistent with each other but cannot be derived from each other.

7. Newton's second law is cause and effect relation. Force is the cause and acceleration is the effect. Conventionally, the effect should be written on the left and cause on the right hand side of the equation. So the correct way of writing Newton's second law is $m\vec{a} = \vec{F}$ or $\frac{d\vec{p}}{dt} = \vec{F}$

3.3 APPLICATION OF NEWTON'S LAWS

3.3.1 Free Body Diagram

Free body diagram is a simple tool to analyse the motion of the object using Newton's laws.

The following systematic steps are followed for developing the free body diagram:

1. Identify the forces acting on the object.
2. Represent the object as a point.
3. Draw the vectors representing the forces acting on the object.

When we draw the free body diagram for an object or a system, the forces exerted by the object should not be included in the free body diagram.

EXAMPLE 3.1

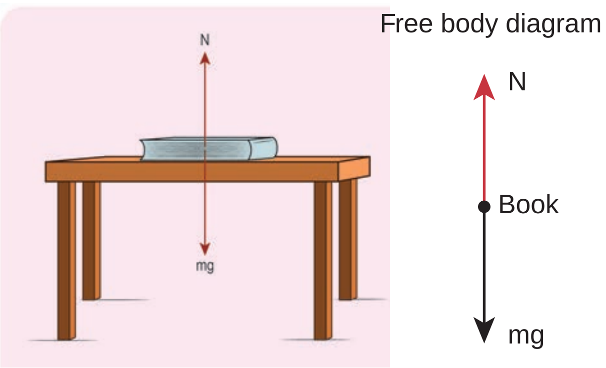
A book of mass m is at rest on the table. (1) What are the forces acting on the book? (2) What are the forces exerted by the book? (3) Draw the free body diagram for the book.

Solution

(1) There are two forces acting on the book.

   (i) Gravitational force (mg) acting downwards on the book

   (ii) Normal contact force (N) exerted by the surface of the table on the book. It acts upwards as shown in the figure.



Note In the free body diagram, as the magnitudes of the normal force and the gravitational force are same, the lengths of both these vectors are also same.

(2) According to Newton's third law, there are two reaction forces exerted by the book.

(i) The book exerts an equal and opposite force (mg) on the Earth which acts upwards.

(ii) The book exerts a force which is equal and opposite to normal force on the surface of the table (N) acting downwards.

Note It is to be emphasized that while applying Newton's third law it is wrong to conclude that the book on the table is at rest due to the downward gravitational force exerted by the Earth and the equal and opposite reacting normal force exerted by the table on the book. Action and reaction forces never act on the same body.

(3) The free body diagram of the book is shown in the figure.

Note Even though the force applied on both the objects is the same, acceleration experienced by each object differs. The acceleration is inversely proportional to mass. For the same force, the heavier mass experiences lesser acceleration and the lighter mass experiences greater acceleration.

When an apple falls, it experiences Earth's gravitational force. According to Newton's third law, the apple exerts equal and opposite force on the Earth. Even though both the apple and Earth experience the same force, their acceleration is different. The mass of Earth is enormous compared to that of an apple. So an apple experiences larger acceleration and the Earth experiences almost negligible acceleration. Due to the negligible acceleration, Earth appears to be stationary when an apple falls.

EXAMPLE 3.2

If two objects of masses 2.5 kg and 100 kg experience the same force 5 N, what is the acceleration experienced by each of them?

Solution

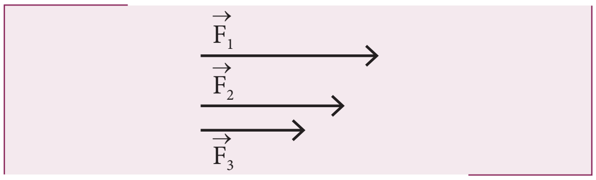
From Newton's second law (in magnitude form), F = ma

For the object of mass 2.5 kg, the acceleration is $a = \frac{F}{m} = \frac{5}{2.5} = 2\ m\ s^{-2}$

For the object of mass 100 kg, the acceleration is $a = \frac{F}{m} = \frac{5}{100} = 0.05\ m\ s^{-2}$

EXAMPLE 3.3

Which is the greatest force among the three force $\vec{F}_1, \vec{F}_2, \vec{F}_3$ shown below



Solution

Force is a vector and magnitude of the vector is represented by the length of the vector. Here $\vec{F}_1$ has greater length compared to other two. So $\vec{F}_1$ is largest of the three.

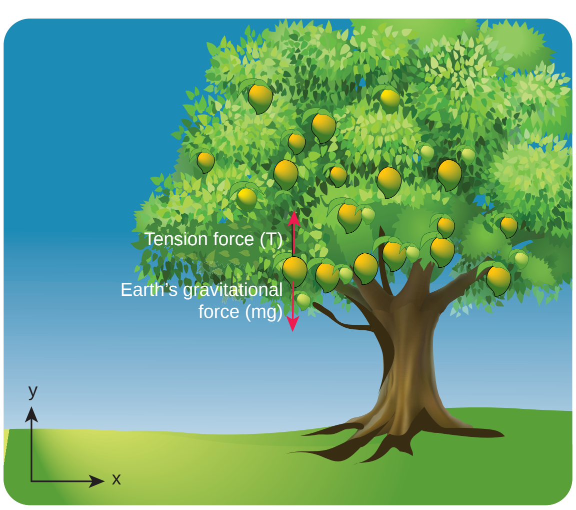
EXAMPLE 3.4

Apply Newton's second law to a mango hanging from a tree. (Mass of the mango is 400 gm)

Solution

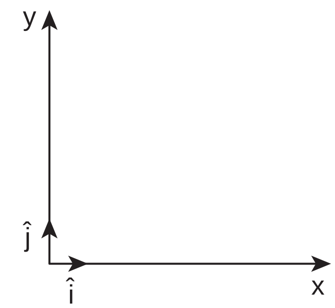
Note: Before applying Newton's laws, the following steps have to be followed:

1) Choose a suitable inertial coordinate system to analyse the problem. For most of the cases we can take Earth as an inertial coordinate system.
2) Identify the system to which Newton's laws need to be applied. The system can be a single object or more than one object.
3) Draw the free body diagram.
4) Once the forces acting on the system are identified, and the free body diagram is drawn, apply Newton's second law. In the left hand side of the equation, write the forces acting on the system in vector notation and equate it to the right hand side of equation which is the product of mass and acceleration. Here, acceleration should also be in vector notation.
5) If acceleration is given, the force can be calculated. If the force is given, acceleration can be calculated.



By following the above steps:

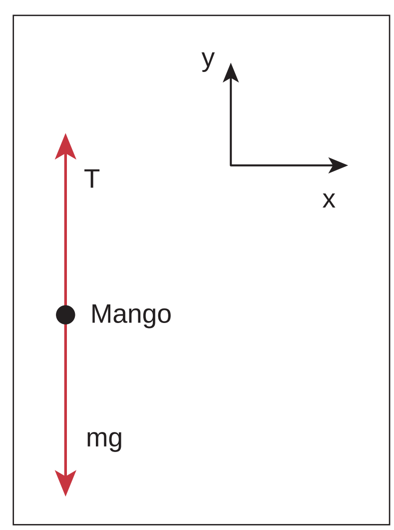
We fix the inertial coordinate system on the ground as shown in the figure.



The forces acting on the mango are

i) Gravitational force exerted by the Earth on the mango acting downward along negative y axis
ii) Tension (in the cord attached to the mango) acts upward along positive y axis.

The free body diagram for the mango is shown in the figure



$$\vec{F}_g = mg\left(-\hat{j}\right) = -mg\hat{j}$$

Here, mg is the magnitude of the gravitational force and $\left(-\hat{j}\right)$ represents the unit vector in negative y direction

$$\vec{T} = T\hat{j}$$

Here T is the magnitude of the tension force and $(\hat{j})$ represents the unit vector in positive y direction

$$\vec{F}_{net} = \vec{F}_g + \vec{T} = -mg\hat{j} + T\hat{j} = (T - mg)\hat{j}$$

From Newton's second law $\vec{F}_{net} = m\vec{a}$

Since the mango is at rest with respect to us (inertial coordinate system) the acceleration is zero $(\vec{a} = 0)$.

So $\vec{F}_{net} = m\vec{a} = 0$

$$(T - mg)\hat{j} = 0$$

By comparing the components on both sides of the above equation, we get $T - mg = 0$

So the tension force acting on the mango is given by $T = mg$

Mass of the mango m = $400g$ and $g = 9.8\ m\ s^{-2}$

Tension acting on the mango is $T = 0.4 \times 9.8 = 3.92$ N

EXAMPLE 3.5

A person rides a bike with a constant velocity $\vec{v}$ with respect to ground and another biker accelerates with acceleration $\vec{a}$ with respect to ground. Who can apply Newton's second law with respect to a stationary observer on the ground?

Solution

Second biker cannot apply Newton's second law, because he is moving with acceleration $\vec{a}$ with respect to Earth (he is not in inertial frame). But the first biker can apply Newton's second law because he is moving at constant velocity with respect to Earth (he is in inertial frame).

EXAMPLE 3.6

The position vector of a particle is given by $\vec{r} = 3t\hat{i} + 5t^2\hat{j} + 7\hat{k}$. Find the direction in which the particle experiences net force?

Solution

Velocity of the particle,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(3t)\hat{i} + \frac{d}{dt}(5t^2)\hat{j} + \frac{d}{dt}(7)\hat{k}$$

$$\frac{d\vec{r}}{dt} = 3\hat{i} + 10t\hat{j}$$

Acceleration of the particle

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = 10\hat{j}$$

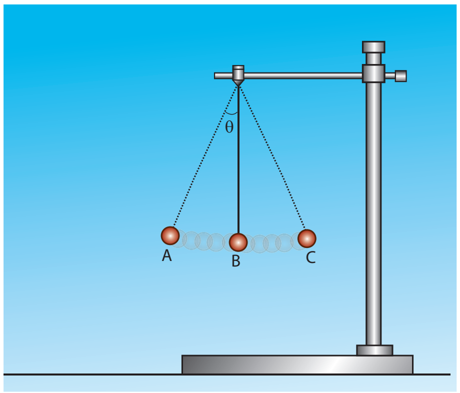
Here, the particle has acceleration only along positive y direction. According to Newton's second law, net force must also act along positive y direction. In addition, the particle has constant velocity in positive x direction and no velocity in z direction. Hence, there are no net force along x or z direction.

EXAMPLE 3.7

Consider a bob attached to a string, hanging from a stand. It oscillates as shown in the figure.

a) Identify the forces that act on the bob?

b) What is the acceleration experienced by the bob?

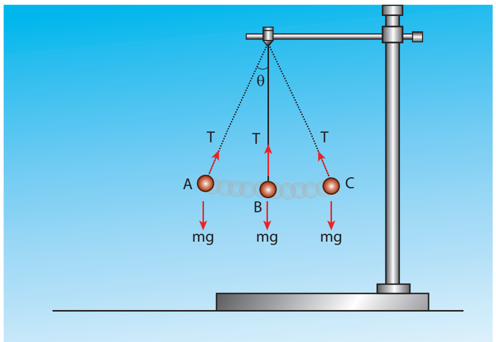


Solution

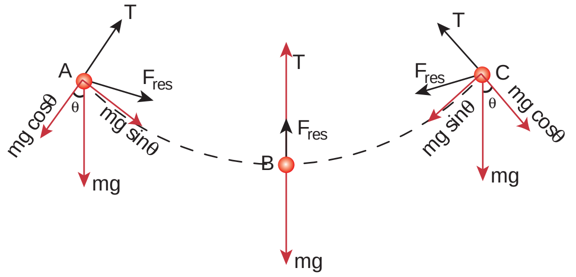
Two forces act on the bob.

(i) Gravitational force (mg) acting downwards

(ii) Tension (T) exerted by the string on the bob, whose position determines the direction of T as shown in figure.



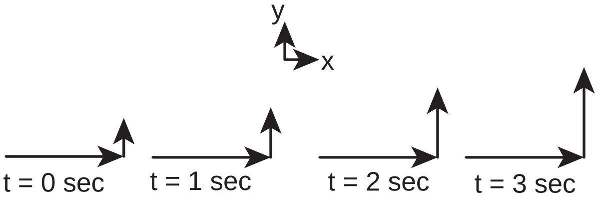
The bob is moving in a circular arc as shown in the above figure. Hence it has centripetal acceleration. At a point A and C, the bob comes to rest momentarily and then its velocity increases when it moves towards point B. Hence, there is a tangential acceleration along the arc. The gravitational force can be resolved into two components (mg cosθ, mg sinθ) as shown below



Note Note that the bob does not move in the direction of the resultant force. At the points A and C, tension T = mg cosθ. At all other points, tension T is greater than mg cosθ, since it has non zero centripetal acceleration. At point B, the resultant force acts upward along the string. It is an example of a non uniform circular motion because the bob has both the centripetal and tangential accelerations.

EXAMPLE 3.8

The velocity of a particle moving in a plane is given by the following diagram. Find out the direction of force acting on the particle?



Solution

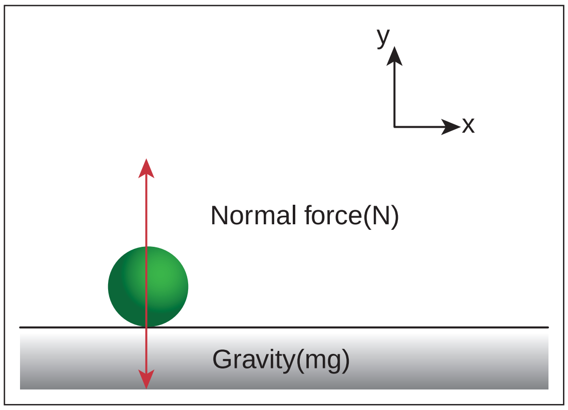
The velocity of the particle is $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$. As shown in the figure, the particle is moving in the xy plane, there is no motion in the z direction. So velocity in the z direction is zero ($v_z = 0$). The velocity of the particle has x component (v_x) and y component (v_y). From figure, as time increases from t = 0 sec to t = 3 sec, the length of the vector in y direction is changing (increasing). It means y component of velocity $\left(v_y\right)$ is increasing with respect to time. According to Newton's second law, if velocity changes with respect to time then there must be acceleration. In this case, the particle has acceleration in the y direction since the y component of velocity changes. So the particle experiences force in the y direction. The length of the vector in x direction does not change. It means that the particle has constant velocity in the x direction. So no force or zero net force acts in the x direction.

EXAMPLE 3.9

Apply Newton's second law for an object at rest on Earth and analyse the result.

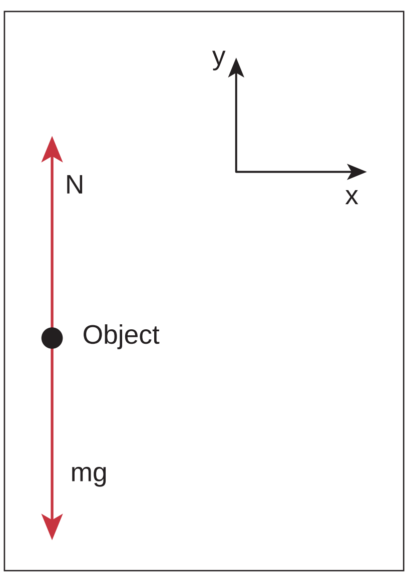
Solution

The object is at rest with respect to Earth (inertial coordinate system). There are two forces that act on the object.



i) Gravity acting downward (negative y-direction)
ii) Normal force by the surface of the Earth acting upward (positive y-direction)

The free body diagram for this object is



$$\vec{F}_g = -mg\hat{j}$$
$$\vec{N} = N\hat{j}$$

Net force $\vec{F}_{net} = -mg\hat{j} + N\hat{j}$

But there is no acceleration on the ball. So $\vec{a} = 0$. By applying Newton's second law $(\vec{F}_{net} = m\vec{a})$

Since $\vec{a} = 0, \vec{F}_{net} = -mg\hat{j} + N\hat{j}$

$$(-mg + N)\hat{j} = 0$$

By comparing the components on both sides of the equation, we get

$$-mg + N = 0$$
$$N = mg$$

We can conclude that if the object is at rest, the magnitude of normal force is exactly equal to the magnitude of gravity.

EXAMPLE 3.10

A particle of mass 2 kg experiences two forces, $\vec{F}_1 = 5\hat{i} + 8\hat{j} + 7\hat{k}$ and $\vec{F}_2 = 3\hat{i} - 4\hat{j} + 3\hat{k}$. What is the acceleration of the particle?

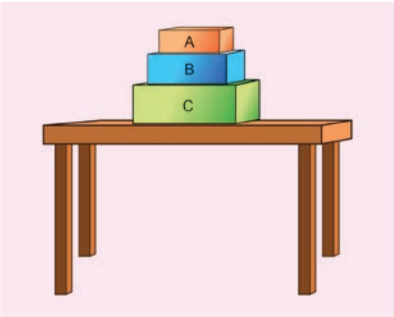
Solution

We use Newton's second law, $\vec{F}_{net} = m\vec{a}$ where $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$. From the above equations the acceleration is $\vec{a} = \frac{\vec{F}_{net}}{m}$, where

$$\vec{F}_{net} = (5+3)\hat{i} + (8-4)\hat{j} + (7+3)\hat{k}$$
$$\vec{F}_{net} = 8\hat{i} + 4\hat{j} + 10\hat{k}$$
$$\vec{a} = \left(\frac{8}{2}\right)\hat{i} + \left(\frac{4}{2}\right)\hat{j} + \left(\frac{10}{2}\right)\hat{k}$$
$$\vec{a} = 4\hat{i} + 2\hat{j} + 5\hat{k}$$

EXAMPLE 3.11

Identify the forces acting on blocks A, B and C shown in the figure.



UNIT-3(XI-Physics_Vol-1).indd 119 05-01-2022 22:17:37

Solution

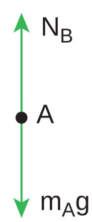
Forces on block A:

(i) Downward gravitational force exerted by the Earth ($m_A g$)

(ii) Upward normal force exerted by block B (N_B)

The free body diagram for block A is as shown in the following picture.

Force on block A



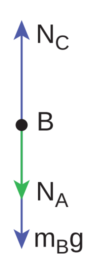
Forces on block B :

(i) Downward gravitational force exerted by Earth ($m_B g$)

(ii) Downward force exerted by block A (N_A)

(iii) Upward normal force exerted by block C (N_C)

Force on block B



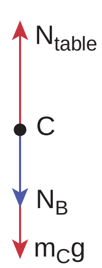
Forces onblock C:

(i) Downward gravitational force exerted by Earth ($m_C g$)

(ii) Downward force exerted by block B (N_B)

(iii) Upward force exerted by the table (N_{table})

Force on block C



EXAMPLE 3.12

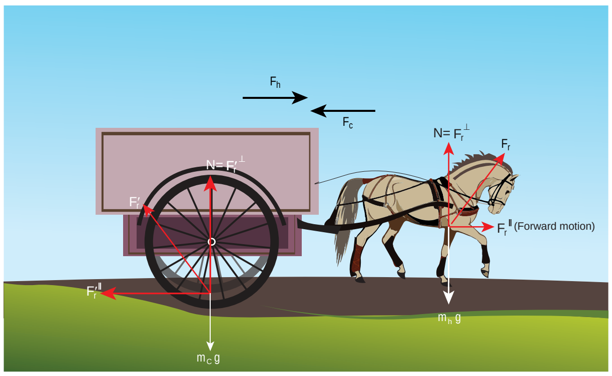
Consider a horse attached to the cart which is initially at rest. If the horse starts walking forward, the cart also accelerates in the forward direction. If the horse pulls the cart with force F_h in forward direction, then according to Newton's third law, the cart also pulls the horse by equivalent opposite force $F_c = F_h$ in backward direction. Then total force on 'cart+horse' is zero. Why is it then the 'cart+horse' accelerates and moves forward?

Solution

This paradox arises due to wrong application of Newton's second and third laws. Before applying Newton's laws, we should decide 'what is the system?'. Once we identify the 'system', then it is possible to identify all the forces acting on the system. We should not consider the force exerted by the system. If there is an unbalanced force acting on the system, then it should have acceleration in the direction of the resultant force. By following these steps we will analyse the horse and cart motion.

If we decide on the cart+horse as a 'system', then we should not consider the force exerted by the horse on the cart or the force exerted by cart on the horse. Both are internal forces acting on each other. According to Newton's third law, total internal force acting on the system is zero and it cannot accelerate the system. The acceleration of the system is caused by some external force. In this case, the force exerted by the road on the system is the external force acting on the system. It is wrong to conclude that the total force acting on the system (cart+horse) is zero without including all the forces acting on the system. The road is pushing the horse and cart forward with acceleration. As there is an external force acting on the system, Newton's second law has to be applied and not Newton's third law.

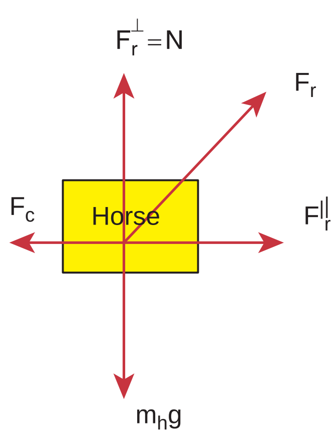
The following figures illustrates this.



If we consider the horse as the 'system', then there are three forces acting on the horse.

(i) Downward gravitational force $(m_h g)$
(ii) Force exerted by the road (F_r)
(iii) Backward force exerted by the cart (F_c)

It is shown in the following figure.



F_r – Force exerted by the road on the horse

F_c – Force exerted by the cart on the horse

$F_r^{\perp}$ – Perpendicular component of $F_r = N$

$F_r^{\parallel}$ – Parallel component of F_r which is reason for forward movement

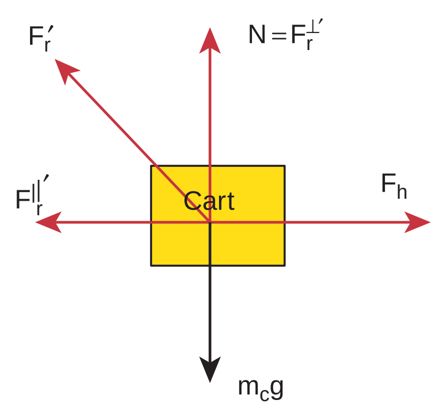
The force exerted by the road can be resolved into parallel and perpendicular components. The perpendicular component balances the downward gravitational force. There is parallel component along the forward direction. It is greater than the backward force (F_c). So there is net force along the forward direction which causes the forward movement of the horse.

If we take the cart as the system, then there are three forces acting on the cart.

(i) Downward gravitational force $(m_c g)$
(ii) Force exerted by the road (F_r')
(iii) Force exerted by the horse (F_h)

It is shown in the figure

Force on the cart



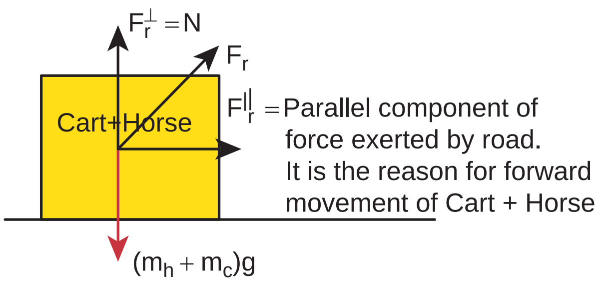
The force exerted by the road $(\vec{F_r'})$ can be resolved into parallel and perpendicular components. The perpendicular component cancels the downward gravity $(m_c g)$. Parallel component acts backwards and the force exerted by the horse $(\vec{F_h})$ acts forward. Force $(\vec{F_h})$ is greater than the parallel component acting in the opposite direction. So there is an overall unbalanced force in the forward direction which causes the cart to accelerate forward.

If we take the cart+horse as a system, then there are two forces acting on the system.

(i) Downward gravitational force $(m_h + m_c)g$

(ii) The force exerted by the road (F_r) on the system.

It is shown in the following figure.



(iii) In this case the force exerted by the road (F_r) on the system (cart+horse) is resolved in to parallel and perpendicular components. The perpendicular component is the normal force which cancels the downward gravitational force $(m_h + m_c)g$. The parallel component of the force is not balanced, hence the system (cart+horse) accelerates and moves forward due to this force.

EXAMPLE 3.13

The position of the particle is represented by $y = ut - \frac{1}{2}gt^2$.

a) What is the force acting on the particle?
b) What is the momentum of the particle?

Solution

To find the force, we need to find the acceleration experienced by the particle.

The acceleration is given by $a = \frac{d^2y}{dt^2}$

$$\text{(or)} \quad a = \frac{dv}{dt}$$

Here

v =velocity of the particle in y direction

$$v = \frac{dy}{dt} = u - gt$$

The momentum of the particle = mv = m (u-gt).

$$a = \frac{dv}{dt} = -g$$

The force acting on the object is given by $F = ma = -mg$

The negative sign implies that the force is acting on the negative y direction. This is exactly the force that acts on the object in projectile motion.

3.3.2 Particle Moving in an Inclined Plane

When an object of mass m slides on a frictionless surface inclined at an angle θ as shown in the Figure 3.12, the forces acting on it decides the

a) acceleration of the object
b) speed of the object when it reaches the bottom

The force acting on the object is

(i) Downward gravitational force (mg)
(ii) Normal force perpendicular to inclined surface (N)

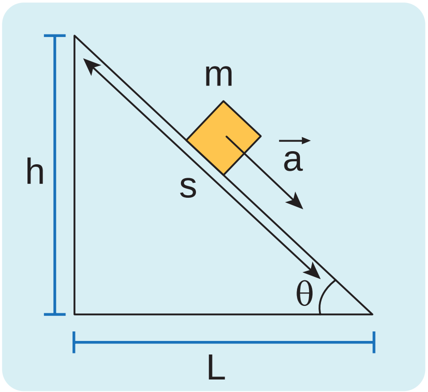


Figure 3.12 Object moving in an inclined plane

To draw the free body diagram, the block is assumed to be a point mass (Figure 3.13 (a)). Since the motion is on the inclined surface, we have to choose the coordinate system parallel to the inclined surface as shown in Figure 3.13 (b).

The gravitational force mg is resolved in to parallel component $mg\sin\theta$ along the inclined plane and perpendicular component $mg\cos\theta$ perpendicular to the inclined surface (Figure 3.13 (b)).

Note that the angle made by the gravitational force (mg) with the perpendicular to the surface is equal to the angle of inclination θ as shown in Figure 3.13 (c).

There is no motion(acceleration) along the y axis. Applying Newton's second law in the y direction

$$-mg\cos\theta\hat{j} + N\hat{j} = 0\,(No\ acceleration)$$

By comparing the components on both sides, $N - mg\cos\theta = 0$

$$N = mg\cos\theta$$

The magnitude of normal force (N) exerted by the surface is equivalent to $mg\cos\theta$.

The object slides (with an acceleration) along the x direction. Applying Newton's second law in the x direction

$$mg\sin\theta\hat{i} = ma\hat{i}$$

By comparing the components on both sides, we can equate

$$mg\sin\theta = ma$$

The acceleration of the sliding object is

$$a = g\ sin\theta$$

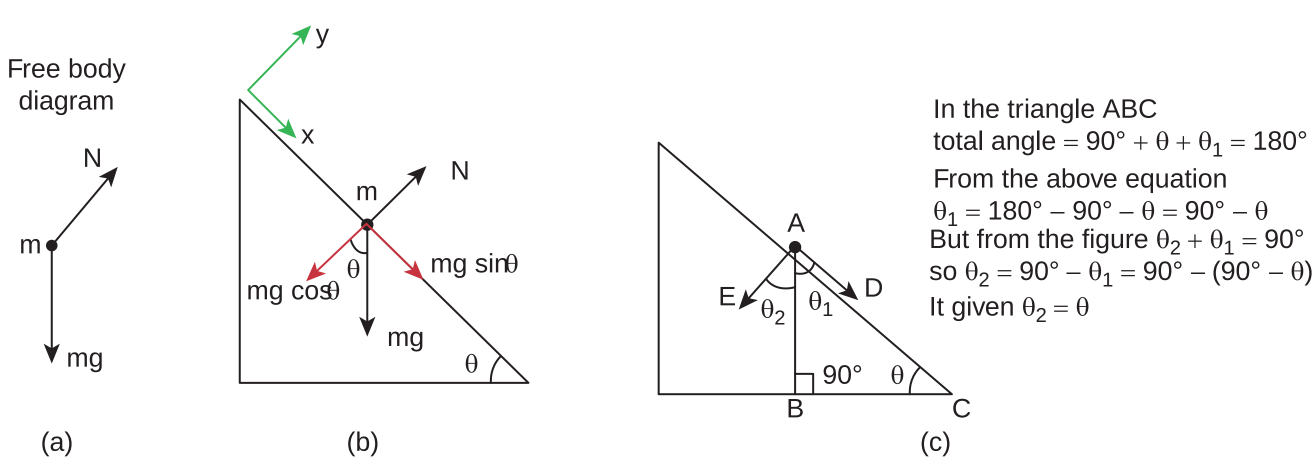


Figure 3.13 (a) Free body diagram, (b) mg resolved into parallel and perpendicular components (c) The angle θ_2 is equal to θ

Note that the acceleration depends on the angle of inclination θ. If the angle θ is 90 degree, the block will move vertically with acceleration $a = g$.

Newton's kinematic equation is used to find the speed of the object when it reaches the bottom. The acceleration is constant throughout the motion.

$$v^2 = u^2 + 2as \text{ along the x direction} \quad (3.3)$$

The acceleration a is equal to g sinθ. The initial speed (u) is equal to zero as it starts from rest. Here s is the length of the inclined surface.

The speed (v) when it reaches the bottom is (using equation (3.3))

$$v = \sqrt{2sg\ \sin\theta} \quad (3.4)$$

Note Here we choose the coordinate system along the inclined plane. Even if we choose the coordinate system parallel to the horizontal surface, we will get the same result. But the mathematics will be quite complicated. Choosing a suitable inertial coordinate system for the given problem is very important.

3.3.3 Two Bodies in Contact on a Horizontal Surface

Consider two blocks of masses m_1 and m_2 ($m_1 > m_2$) kept in contact with each other on a smooth, horizontal frictionless surface as shown in Figure 3.14.

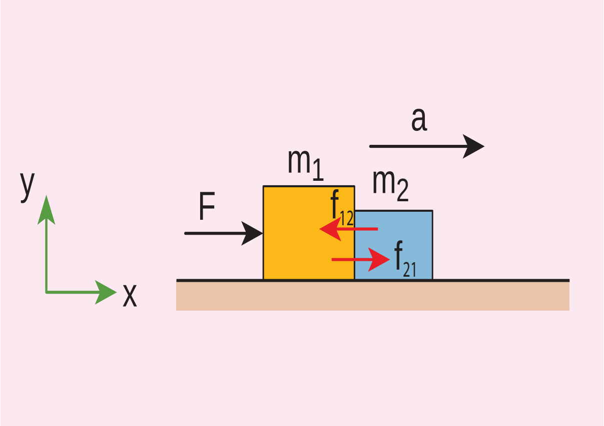


Figure 3.14 (a) Two blocks of masses m_1 and m_2 ($m_1 > m_2$) kept in contact with each other on a smooth, horizontal frictionless surface

By the application of a horizontal force F, both the blocks are set into motion with acceleration 'a' simultaneously in the direction of the force F.

To find the acceleration $\vec{a}$, Newton's second law has to be applied to the system (combined mass $m = m_1 + m_2$)

$$\vec{F} = m\vec{a}$$

If we choose the motion of the two masses along the positive x direction,

$$F\hat{i} = ma\hat{i}$$

By comparing components on both sides of the above equation

$$F = ma \quad \text{where } m = m_1 + m_2$$

The acceleration of the system is given by

$$\therefore a = \frac{F}{m_1 + m_2} \quad (3.5)$$

The force exerted by the block m_1 on m_2 due to its motion is called force of contact $(\vec{f}_{21})$. According to Newton's third law, the block m_2 will exert an equivalent opposite reaction force $(\vec{f}_{12})$ on block m_1.

Figure 3.14 (b) shows the free body diagram of block m_1.

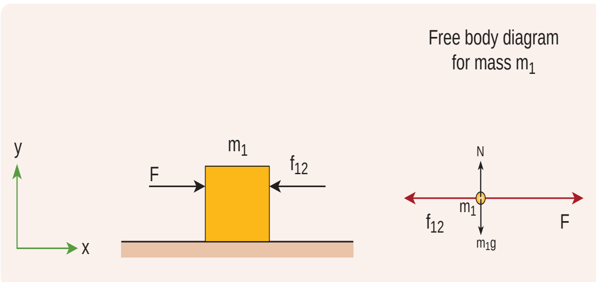


Figure 3.14 (b) Free body diagram of block of mass m_1

$$\therefore F\hat{i} - f_{12}\hat{i} = m_1 a\hat{i}$$

By comparing the components on both sides of the above equation, we get

$$F - f_{12} = m_1 a$$

$$f_{12} = F - m_1 a \qquad (3.6)$$

Substituting the value of acceleration from equation (3.5) in (3.6) we get

$$f_{12} = F - m_1\left(\frac{F}{m_1 + m_2}\right)$$

$$f_{12} = F\left[1 - \frac{m_1}{m_1 + m_2}\right]$$

$$f_{12} = \frac{Fm_2}{m_1 + m_2} \qquad (3.7)$$

Equation (3.7) shows that the magnitude of contact force depends on mass m_2 which provides the reaction force. Note that this force is acting along the negative x direction.

In vector notation, the reaction force on mass m_1 is given by $\vec{f}_{12} = -\dfrac{Fm_2}{m_1 + m_2}\hat{i}$

For mass m_2 there is only one force acting on it in the x direction and it is denoted by $\vec{f}_{21}$. This force is exerted by mass m_1. The free body diagram for mass m_2 is shown in Figure 3.14 (c).

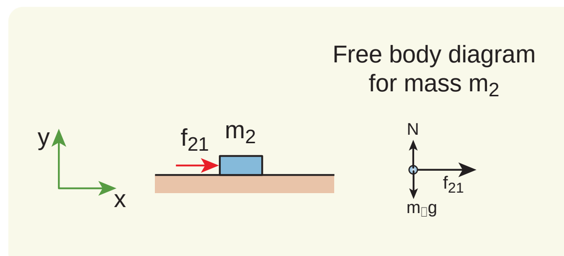


Figure 3.14 (c) Free body diagram of block of mass m_2

Applying Newton's second law for mass m_2

$$f_{21}\hat{i} = m_2 a\hat{i}$$

By comparing the components on both sides of the above equation

$$f_{21} = m_2 a \qquad (3.8)$$

Substituting for acceleration from equation (3.5) in equation (3.8), we get $f_{21} = \dfrac{Fm_2}{m_1 + m_2}$

In this case the magnitude of the contact force is

$f_{21} = \dfrac{Fm_2}{m_1 + m_2}$ The direction of this force is along the positive x direction.

In vector notation, the force acting on mass m_2 exerted by mass m_1 is $\vec{f}_{21} = \frac{Fm_2}{m_1 + m_2}\hat{i}$

Note $\vec{f}_{12} = -\vec{f}_{21}$ which confirms Newton's third law.

3.3.4 Motion of Connected Bodies

When objects are connected by strings and a force F is applied either vertically or horizontally or along an inclined plane, it produces a tension T in the string, which affects the acceleration to an extent. Let us discuss various cases for the same.

Case 1: Vertical motion

Consider two blocks of masses m_1 and m_2 ($m_1 > m_2$) connected by a light and inextensible string that passes over a pulley as shown in Figure 3.15.

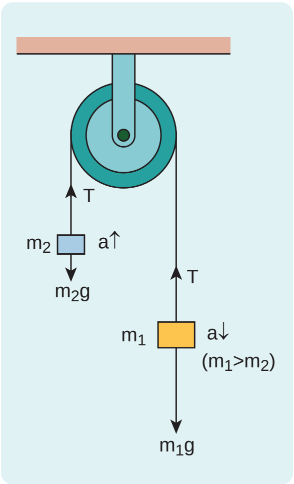


Figure 3.15 Two blocks connected by a string over a pulley

Let the tension in the string be T and acceleration a. When the system is released, both the blocks start moving, m_2 vertically upward and m_1 downward with same acceleration a. The gravitational force m_1g on mass m_1 is used in lifting the mass m_2.

The upward direction is chosen as y direction. The free body diagrams of both masses are shown in Figure 3.16.

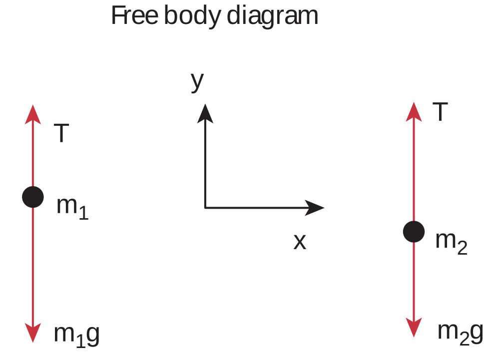


Figure 3.16 Free body diagrams of masses m_1 and m_2

Applying Newton's second law for mass m_2

$$T\hat{j} - m_2 g\hat{j} = m_2 a\hat{j}$$

The left hand side of the above equation is the total force that acts on m_2 and the right hand side is the product of mass and acceleration of m_2 in y direction.

By comparing the components on both sides, we get

$$T - m_2 g = m_2 a \qquad (3.9)$$

Similarly, applying Newton's second law for mass m_1

$$T\hat{j} - m_1 g\hat{j} = -m_1 a\hat{j}$$

As mass m_1 moves downward $(-\hat{j})$, its acceleration is along $(-\hat{j})$

By comparing the components on both sides, we get

$$T - m_1 g = -m_1 a$$

$$m_1 g - T = m_1 a \qquad (3.10)$$

Adding equations (3.9) and (3.10), we get

$$m_1 g - m_2 g = m_1 a + m_2 a$$

$$(m_1 - m_2) g = (m_1 + m_2) a \qquad (3.11)$$

From equation (3.11), the acceleration of both the masses is

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \qquad (3.12)$$

If both the masses are equal ($m_1 = m_2$), from equation (3.12)

$$a = 0$$

This shows that if the masses are equal, there is no acceleration and the system as a whole will be at rest.

To find the tension acting on the string, substitute the acceleration from the equation (3.12) into the equation (3.9).

$$T - m_2 g = m_2 \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

$$T = m_2 g + m_2 \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \qquad (3.13)$$

By taking $m_2 g$ common in the RHS of equation (3.13)

$$T = m_2 g \left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$T = m_2 g \left(\frac{m_1 + m_2 + m_1 - m_2}{m_1 + m_2}\right)$$

$$T = \left(\frac{2 m_1 m_2}{m_1 + m_2}\right) g$$

Equation (3.12) gives only magnitude of acceleration.

For mass m_1, the acceleration vector is given by $\vec{a} = -\left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \hat{j}$

For mass m_2, the acceleration vector is given by $\vec{a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \hat{j}$

Case 2: Horizontal motion

In this case, mass m_2 is kept on a horizontal table and mass m_1 is hanging through a small pulley as shown in Figure 3.17. Assume that there is no friction on the surface.

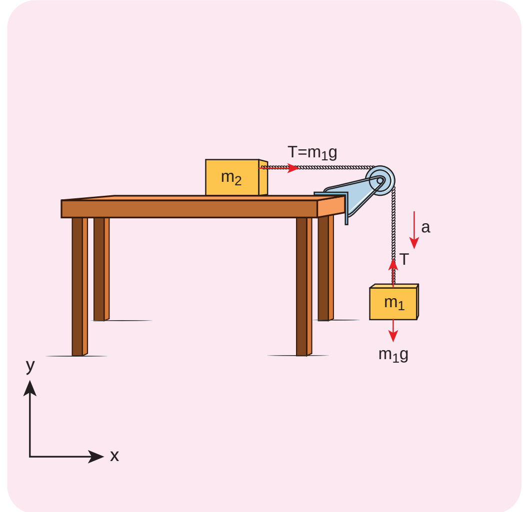


Figure 3.17 Blocks in horizontal motion

As both the blocks are connected to the unstretchable string, if m_1 moves with an acceleration a downward then m_2 also moves with the same acceleration a horizontally.

The forces acting on mass m_2 are

(i) Downward gravitational force (m_2g)
(ii) Upward normal force (N) exerted by the surface
(iii) Horizontal tension (T) exerted by the string

The forces acting on mass m_1 are

(i) Downward gravitational force (m_1g)
(ii) Tension (T) acting upwards

The free body diagrams for both the masses is shown in Figure 3.18.

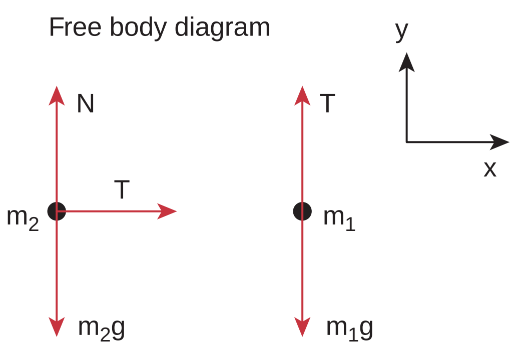


Figure 3.18 Free body diagrams of masses m_1 and m_2

Applying Newton's second law for m_1

$$T\hat{j} - m_1g\hat{j} = -m_1a\hat{j} \text{ (along y direction)}$$

By comparing the components on both sides of the above equation,

$$T - m_1g = -m_1a \qquad (3.14)$$

Applying Newton's second law for m_2

$$T\hat{i} = m_2a\hat{i} \text{ (along x direction)}$$

By comparing the components on both sides of above equation,

$$T = m_2a \qquad (3.15)$$

There is no acceleration along y direction for m_2.

$$N\hat{j} - m_2g\hat{j} = 0$$

By comparing the components on both sides of the above equation

$$N - m_2g = 0$$

$$N = m_2g \qquad (3.16)$$

By substituting equation (3.15) in equation (3.14), we can find the tension T

$$m_2a - m_1g = -m_1a$$

$$m_2a + m_1a = m_1g$$

$$a = \frac{m_1}{m_1 + m_2}g \qquad (3.17)$$

Tension in the string can be obtained by substituting equation (3.17) in equation (3.15)

$$T = \frac{m_1m_2}{m_1 + m_2}g \qquad (3.18)$$

Comparing motion in both cases, it is clear that the tension in the string for horizontal motion is half of the tension for vertical motion for same set of masses and strings.

This result has an important application in industries. The ropes used in conveyor belts (horizontal motion) work for longer duration than those of cranes and lifts (vertical motion).

3.3.5 Concurrent Forces and Lami's Theorem

A collection of forces is said to be concurrent, if the lines of forces act at a common point. Figure 3.19 illustrates concurrent forces.

Concurrent forces need not be in the same plane. If they are in the same plane, they are concurrent as well as coplanar forces.

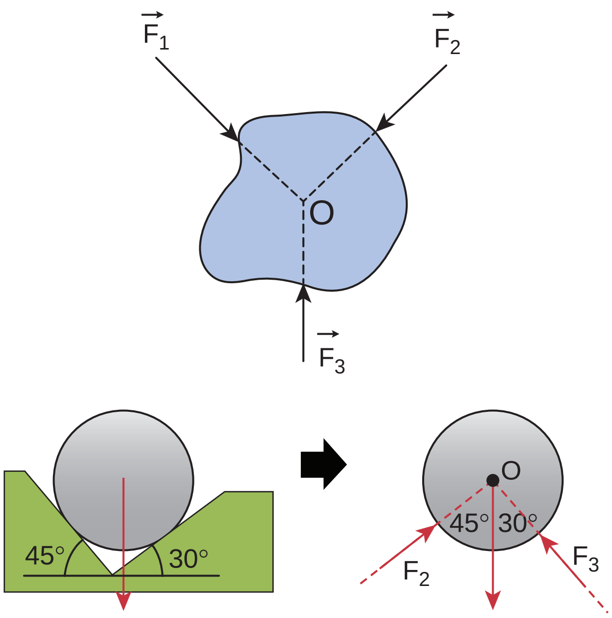


Figure 3.19 Concurrent forces

3.4 LAMI'S THEOREM

If a system of three concurrent and coplanar forces is in equilibrium, then Lami's theorem states that the magnitude of each force of the system is proportional to sine of the angle between the other two forces. The constant of proportionality is same for all three forces.

Let us consider three coplanar and concurrent forces $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ which act at a common point O as shown in Figure 3.20. If the point is at equilibrium, then according to Lami's theorem

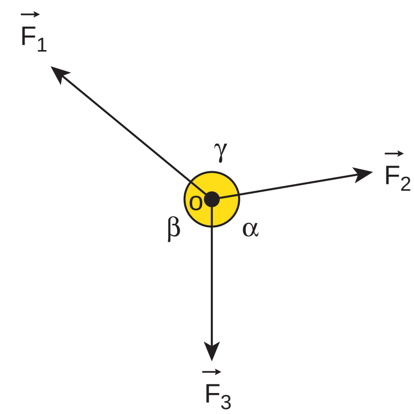


Figure 3.20 Three coplanar and concurrent forces $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ acting at O

$$|\vec{F}_1| \propto \sin\alpha$$

$$|\vec{F}_2| \propto \sin\beta$$

$$|\vec{F}_3| \propto \sin\gamma$$

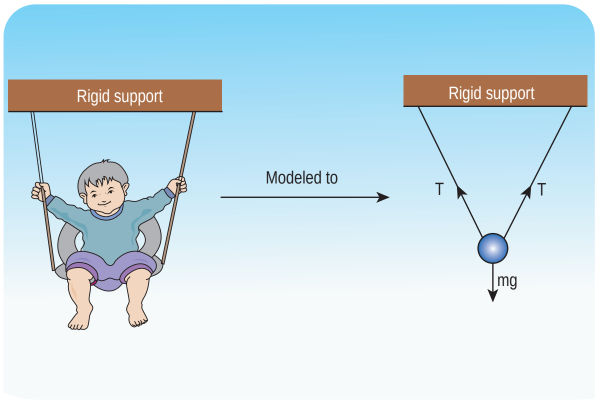
Therefore, $$\frac{|\vec{F}_1|}{\sin\alpha} = \frac{|\vec{F}_2|}{\sin\beta} = \frac{|\vec{F}_3|}{\sin\gamma} \quad (3.19)$$

Lami's theorem is useful to analyse the forces acting on objects which are in static equilibrium.

Application of Lami's Theorem

EXAMPLE 3.14

A baby is playing in a swing which is hanging with the help of two identical chains is at rest. Identify the forces acting on the baby. Apply Lami's theorem and find out the tension acting on the chain.

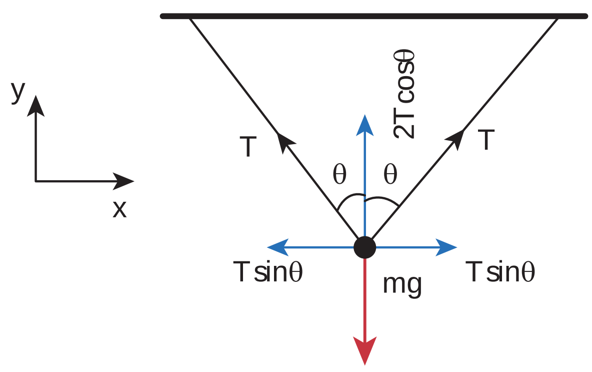


Solution

The baby and the chains are modeled as a particle hung by two strings as shown in the figure. There are three forces acting on the baby.

i) Downward gravitational force along negative y direction (mg)
ii) Tension (T) along the two strings

These three forces are coplanar as well as concurrent as shown in the following figure.



By using Lami's theorem

$$\frac{T}{\sin(180-\theta)} = \frac{T}{\sin(180-\theta)} = \frac{mg}{\sin(2\theta)}$$

Since $\sin(180-\theta) = \sin\theta$ and $\sin(2\theta) = 2\sin\theta\cos\theta$

We get $$\frac{T}{\sin\theta} = \frac{mg}{2\sin\theta\cos\theta}$$

From this, the tension on each string is $T = \frac{mg}{2\cos\theta}$.

Note When $\theta = 0°$, the strings are vertical and the tension on each string is $T = \frac{mg}{2}$

3.5 LAW OF CONSERVATION OF TOTAL LINEAR MOMENTUM

In nature, conservation laws play a very important role. The dynamics of motion of bodies can be analysed very effectively using conservation laws. There are three conservation laws in mechanics. Conservation of total energy, conservation of total linear momentum, and conservation of angular momentum. By combining Newton's second and third laws, we can derive the law of conservation of total linear momentum.

When two particles interact with each other, they exert equal and opposite forces on each other. The particle 1 exerts force $\vec{F}_{21}$ on particle 2 and particle 2 exerts an exactly equal and opposite force $\vec{F}_{12}$ on particle 1, according to Newton's third law.

$$\vec{F}_{21} = -\vec{F}_{12} \quad (3.20)$$

In terms of momentum of particles, the force on each particle (Newton's second law) can be written as

$$\vec{F}_{12} = \frac{d\vec{p}_1}{dt} \text{ and } \vec{F}_{21} = \frac{d\vec{p}_2}{dt}. \quad (3.21)$$

Here $\vec{p}_1$ is the momentum of particle 1 which changes due to the force $\vec{F}_{12}$ exerted by particle 2. Further $\vec{p}_2$ is the momentum of particle 2. This changes due to $\vec{F}_{21}$ exerted by particle 1.

Substitute equation (3.21) in equation (3.20)

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \tag{3.22}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \tag{3.23}$$

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

It implies that $\vec{p}_1 + \vec{p}_2 =$ constant vector (always).

$\vec{p}_1 + \vec{p}_2$ is the total linear momentum of the two particles ($\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2$). It is also called as total linear momentum of the system. Here, the two particles constitute the system. From this result, the law of conservation of linear momentum can be stated as follows.

If there are no external forces acting on the system, then the total linear momentum of the system ($\vec{p}_{tot}$) is always a constant vector. In otherwords, the total linear momentum of the system is conserved in time. Here the word 'conserve' means that $\vec{p}_1$ and $\vec{p}_2$ can vary,in such a way that $\vec{p}_1 + \vec{p}_2$ is a constant vector.

The forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are called the internal forces of the system, because they act only between the two particles. There is no external force acting on the two particles from outside. In such a case the total linear momentum of the system is a constant vector or is conserved.

EXAMPLE 3.15

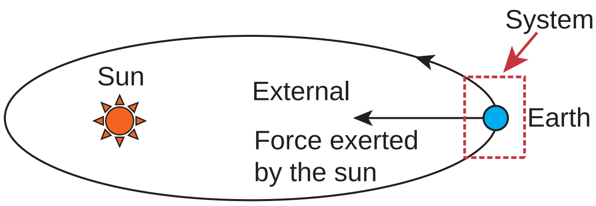
Identify the internal and external forces acting on the following systems.

a) Earth alone as a system
b) Earth and Sun as a system
c) Our body as a system while walking
d) Our body + Earth as a system

Solution

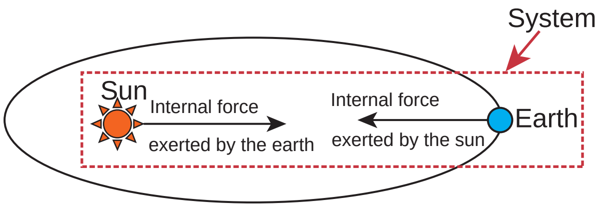
a) Earth alone as a system

Earth orbits the Sun due to gravitational attraction of the Sun. If we consider Earth as a system, then Sun's gravitational force is an external force. If we take the Moon into account, it also exerts an external force on Earth.



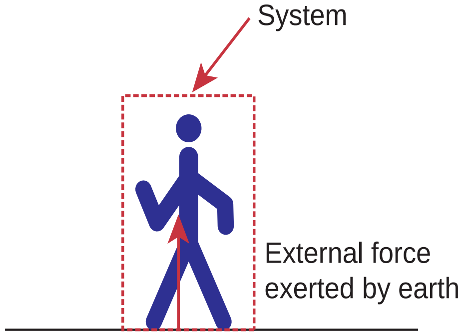
b) (Earth + Sun) as a system

In this case, there are two internal forces which form an action and reaction pair- the gravitational force exerted by the Sun on Earth and gravitational force exerted by the Earth on the Sun.



c) Our body as a system

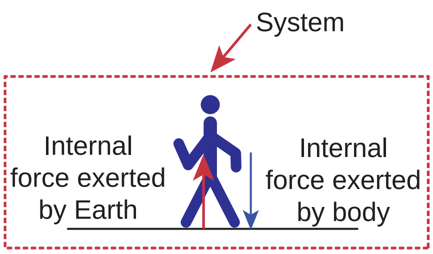
While walking, we exert a force on the Earth and Earth exerts an equal and opposite force on our body. If our body alone is considered as a system, then the force exerted by the Earth on our body is external.



Our body as a system

d) (Our body + Earth) as a system

In this case, there are two internal forces present in the system. One is the force exerted by our body on the Earth and the other is the equal and opposite force exerted by the Earth on our body.

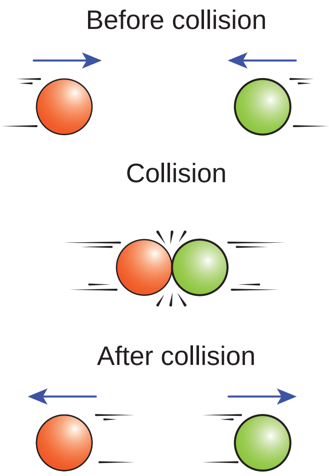


Our body + Earth as a system

Meaning of law of conservation of momentum

1) The Law of conservation of linear momentum is a vector law. It implies that both the magnitude and direction of total linear momentum are constant. In some cases, this total momentum can also be zero.
2) To analyse the motion of a particle, we can either use Newton's second law or the law of conservation of linear momentum. Newton's second law requires us to specify the forces involved in the process. This is difficult to specify in real situations. But conservation of linear momentum does not require any force involved in the process. It is covenient and hence important.

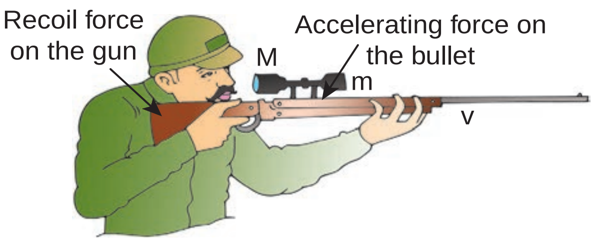
For example, when two particles collide, the forces exerted by these two particles on each other is difficult to specify. But it is easier to apply conservation of linear momentum during the collision process.



Collision of two particles

Examples

- Consider the firing of a gun. Here the system is Gun+bullet. Initially the gun and bullet are at rest, hence the total linear momentum of the system is zero. Let $\vec{p}_1$ be the momentum of the bullet and $\vec{p}_2$ the momentum of the gun before firing. Since initially both are at rest,



$$\vec{p}_1 = 0, \vec{p}_2 = 0.$$

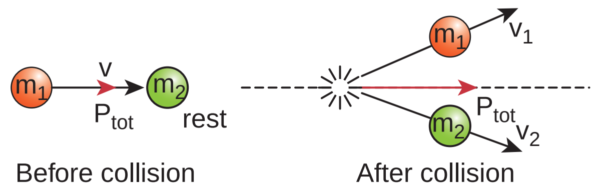
Total momentum before firing the gun is zero, $\vec{p}_1 + \vec{p}_2 = 0$.

According to the law of conservation of linear momentum, total linear momemtum has to be zero after the firing also.

When the gun is fired, a force is exerted by the gun on the bullet in forward direction. Now the momentum of the bullet changes from $\vec{p}_1$ to $\vec{p}_1'$. To conserve the total linear momentum of the system, the momentum of the gun must also change from $\vec{p}_2$ to $\vec{p}_2'$. Due to the conservation of linear momentum, $\vec{p}_1' + \vec{p}_2' = 0$. It implies that $\vec{p}_1' = -\vec{p}_2'$, the momentum of the gun is exactly equal, but in the opposite direction to the momentum of the bullet. This is the reason after firing, the gun suddenly moves backward with the momentum $\left(-\vec{p}_2'\right)$. It is called 'recoil momemtum'. This is an example of conservation of total linear momentum.

S8G1X

- Consider two particles. One is at rest and the other moves towards the first particle (which is at rest). They collide and after collison move in some arbitrary directions. In this case, before collision, the total linear momentum of the system is equal to the initial linear momentum of the moving particle. According to conservation of momentum, the total linear momentum after collision also has to be in the forward direction. The following figure explains this.



A more accurate calculation is covered in section 4.4. It is to be noted that the total momentum vector before and after collison points in the same direction. This simply means that the total linear momentum is constant before and after the collision. At the time of collision, each particle exerts a force on the other. As the two particles are considered as a system, these forces are only internal, and the total linear momentum cannot be altered by internal forces.

3.5.1 Impulse

If a very large force acts on an object for a very short duration, then the force is called impulsive force or impulse.

If a force (F) acts on the object in a very short interval of time (Δt), from Newton's second law in magnitude form

$$Fdt = dp$$

Integrating over time from an initial time t_i to a final time t_f, we get

$$\int_i^f dp = \int_{t_i}^{t_f} F\,dt$$

$$p_f - p_i = \int_{t_i}^{t_f} F\,dt$$

p_i = initial momentum of the object at time t_i

p_f = final momentum of the object at time t_f

$p_f - p_i = \Delta p =$ change in momentum of the object during the time interval $t_f - t_i = \Delta t$.

The integral $\int_{t_i}^{t_f} F\,dt = J$ is called the impulse and it is equal to change in momentum of the object.

If the force is constant over the time interval, then

$$\int_{t_i}^{t_f} F\,dt = F\int_{t_i}^{t_f} dt = F\left(t_f - t_i\right) = F\Delta t$$

$$F\,\Delta t = \Delta p \qquad (3.24)$$

Equation (3.24) is called the 'impulse-momentum equation'.

For a constant force, the impulse is denoted as $J = F\Delta t$ and it is also equal to change in momentum (Δp) of the object over the time interval Δt.

Impulse is a vector quantity and its unit is Ns.

The average force acted on the object over the short interval of time is defined by

$$F_{avg} = \frac{\Delta p}{\Delta t} \qquad (3.25)$$

From equation (3.25), the average force that act on the object is greater if Δt is smaller. Whenever the momentum of the body changes very quickly, the average force becomes larger.

The impulse can also be written in terms of the average force. Since Δp is change in momentum of the object and is equal to impulse (J), we have

$$J = F_{avg}\Delta t \qquad (3.26)$$

The graphical representation of constant force impulse and variable force impulse is given in Figure 3.21.

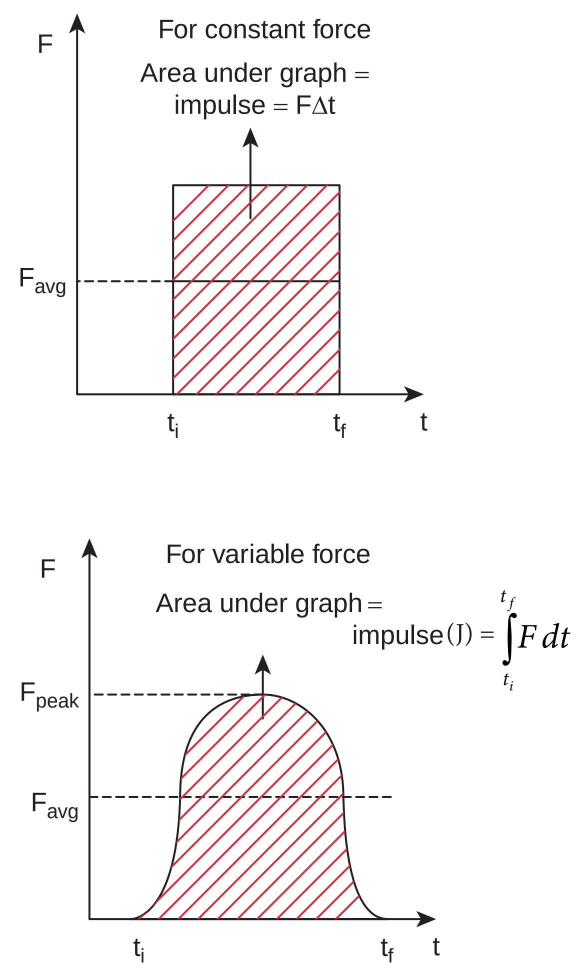


Figure 3.21 Constant force impulse and variable force impulse

Illustration

1. When a cricket player catches the ball, he pulls his hands gradually in the direction of the ball's motion. Why?

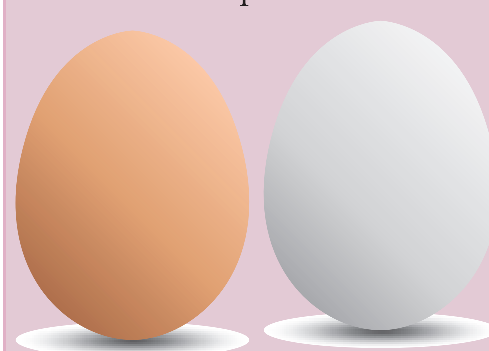
   If he stops his hands soon after catching the ball, the ball comes to rest very quickly. It means that the momentum of the ball is brought to rest very quickly. So the average force acting on the body will be very large. Due to this large average force, the hands will get hurt. To avoid getting hurt, the player brings the ball to rest slowly.

2. When a car meets with an accident, its momentum reduces drastically in a very short time. This is very dangerous for the passengers inside the car since they will experience a large force. To prevent this fatal shock, cars are designed with air bags in such a way that when the car meets with an accident, the momentum of the passengers will reduce slowly so that the average force acting on them will be smaller.

3. The shock absorbers in two wheelers play the same role as airbags in the car. When there is a bump on the road, a sudden force is transferred to the vehicle. The shock absorber prolongs the period of transfer of force on to the body of the rider. Vehicles without shock absorbers will harm the body due to this reason.

4. Jumping on a concrete cemented floor is more dangerous than jumping on the sand. Sand brings the body to rest slowly than the concrete floor, so that the average force experienced by the body will be lesser.

Impulse



If an egg is thrown, can you catch the egg safely without breaking it? How?

EXAMPLE 3.16

An object of mass 10 kg moving with a speed of 15 $m\ s^{-1}$ hits the wall and comes to rest within

a) 0.03 second
b) 10 second

Calculate the impulse and average force acting on the object in both the cases.

Solution

Initial momentum of the object $p_i = 10 \times 15 = 150\ k\ gm\ s^{-1}$

Final momentum of the object $p_f = 0$

$$\Delta p = 150 - 0 = 150\ kg\ m s^{-1}$$

(a) Impulse $J = \Delta p = 150\,N\,s.$

(b) Impulse $J = \Delta p = 150\,N\,s$

(a) Average force $F_{avg} = \dfrac{\Delta p}{\Delta t} = \dfrac{150}{0.03} = 5000\,N$

(b) Average force $F_{avg} = \dfrac{150}{10} = 15\,N$

We see that, impulse is the same in both cases, but the average force is different.

3.6 FRICTION

3.6.1 Introduction

If a very gentle force in the horizontal direction is given to an object at rest on the table, it does not move. It is because of the opposing force exerted by the surface on the object which resists its motion. This force is called the *frictional force which always opposes the relative motion between an object and the surface where it is placed.* If the force applied is increased, the object moves after a certain limit.

Relative motion: when a force parallel to the surface is applied on the object, the force tries to move the object with respect to the surface. This 'relative motion' is opposed

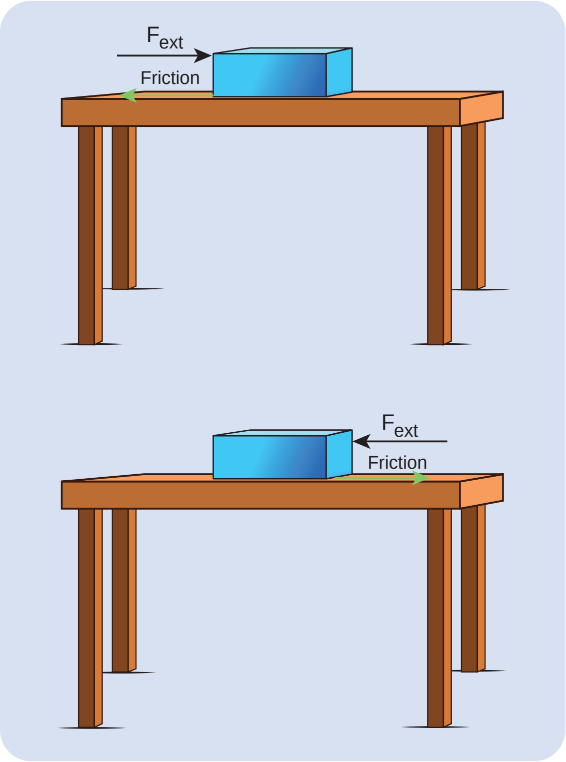


Figure 3.22 Frictional force

by the surface by exerting a frictional force on the object in a direction opposite to applied force. Frictional force always acts on the object parallel to the surface on which the object is placed. There are two kinds of friction namely 1) Static friction and 2) Kinetic friction.

3.6.2 Static Friction ($\vec{f}_s$)

Static friction is the force which opposes the initiation of motion of an object on the surface. When the object is at rest on the surface, only two forces act on it. They are the downward gravitational force and upward normal force. The resultant of these two forces on the object is zero. As a result the object is at rest as shown in Figure 3.23.

If some external force F_{ext} is applied on the object parallel to the surface on which the object is at rest, the surface exerts exactly an equal and opposite force on the object to resist its motion and tries to keep the object at rest. It implies that external force and frictional force are exactly equal and opposite. Therefore, no motion parallel to the surface takes place. But if the external force is increased above a particular limit, the surface cannot provide sufficient opposing frictional force to balance the external force on the object. Then the object starts to slide. This is the maximal static friction that can be exerted by the surface. Experimentally, it is found that the magnitude of static frictional force f_s satisfies the following empirical relation.

$$0 \leq f_s \leq \mu_s N, \qquad (3.27)$$

where μ_s is the coefficient of static friction. It depends on the nature of the surfaces in contact. N is normal force exerted by the surface on the body and sometimes it is equal to mg. But it need not be equal to *mg* always.

Equation (3.27) implies that the force of static friction can take any value from zero to $\mu_s N$.

If the object is at rest and no external force is applied on the object, the static friction acting on the object is zero ($f_s = 0$).

If the object is at rest, and there is an external force applied parallel to the surface, then the force of static friction acting on the object is exactly equal to the external force applied on the object ($f_s = F_{ext}$). But still the static friction f_s is less than $\mu_s N$.

When object begins to slide, the static friction (f_s) acting on the object attains maximum,

The static and kinetic frictions (which we discuss later) depend on the normal force acting on the object. If the object is pressed hard on the surface then the normal force acting on the object will increase. As a consequence it is more difficult to move the object. This is shown in Figure 3.23 (a) and (b). The static friction does not depend upon the area of contact.

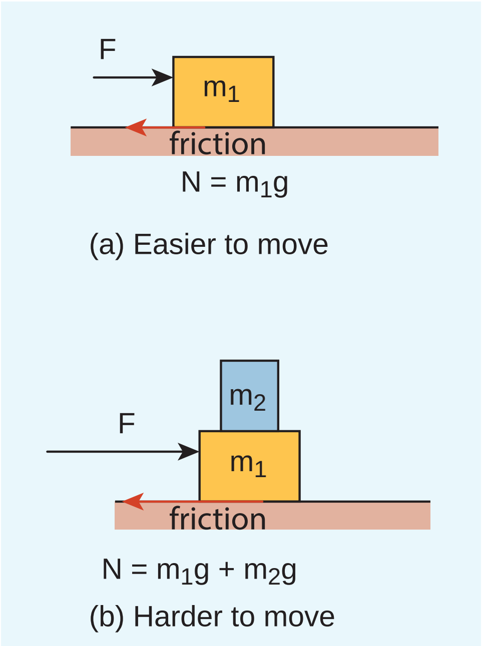


Fig 3.23 Static friction and kinetic friction (a) Easier to move (b) Harder to move

EXAMPLE 3.17

Consider an object of mass 2 kg resting on the floor. The coefficient of static friction between the object and the floor is $\mu_s = 0.8$. What force must be applied on the object to move it?

Solution

Since the object is at rest, the gravitational force experienced by an object is balanced by normal force exerted by floor.

$$N = mg$$

The maximum static frictional force $f_s^{max} = \mu_s N = \mu_s mg$

$$f_s^{max} = 0.8 \times 2 \times 9.8 = 15.68\,N$$

Therefore to move the object the external force should be greater than maximum static friction.

$$F_{ext} > 15.68\,N$$

EXAMPLE 3.18

Consider an object of mass 50 kg at rest on the floor. A Force of 5 N is applied on the object but it does not move. What is the frictional force that acts on the object?

Solution

When the object is at rest, the external force and the static frictional force are equal and opposite.

The magnitudes of these two forces are equal, $f_s = F_{ext}$

Therefore, the static frictional force acting on the object is

$$f_s = 5N.$$

The direction of this frictional force is opposite to the direction of F_{ext}.

EXAMPLE 3.19

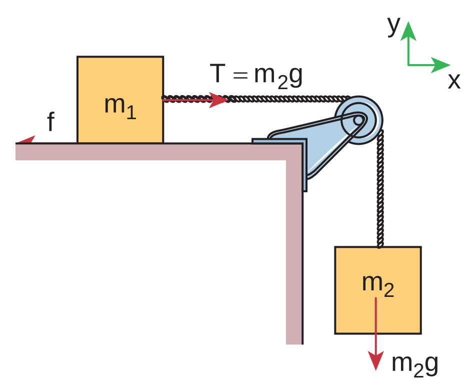
Two bodies of masses 7 kg and 5 kg are connected by a light string passing over a smooth pulley at the edge of the table as shown in the figure. The coefficient of static friction between the surfaces (body and table) is 0.9. Will the mass m_1 = 7 kg on the surface move? If not what value of m_2 should be used so that mass 7 kg begins to slide on the table?

Solution

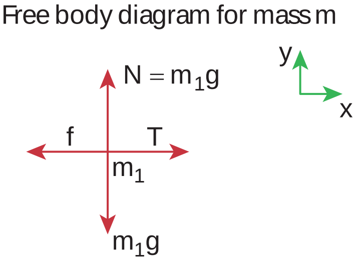
As shown in the figure, there are four forces acting on the mass m_1

a) Downward gravitational force along the negative y-axis (m_1g)
b) Upward normal force along the positive y axis (N)
c) Tension force due to mass m_2 along the positive x axis
d) Frictional force along the negative x axis

Since the mass m_1 has no vertical motion, $m_1g = N$



Free body diagram for mass m_1



To determine whether the mass m_1 moves on the surface, calculate the maximum static friction exerted by the table on the mass m_1. If the tension on the mass m_1 is equal to or greater than this maximum static friction, the object will move.

$$f_s^{max} = \mu_s N = \mu_s m_1 g$$

$$f_s^{max} = 0.9 \times 7 \times 9.8 = 61.74\ N$$

The tension $T = m_2 g = 5 \times 9.8 = 49\ N$

$$T < f_s^{max}$$

The tension acting on the mass m_1 is less than the maximum static friction. So the mass m_1 will not move.

To move the mass m_1, $T > f_s^{max}$ where $T = m_2 g$

$$m_2 = \frac{\mu_s m_1 g}{g} = \mu_s m_1$$

$$m_2 = 0.9 \times 7 = 6.3\ kg$$

If the mass m_2 is greater than 6.3 kg then the mass m_1 will begin to slide. Note that if there is no friction on the surface, the mass m_1 will move even when m_2 is just 1 kg.

The values of coefficient of static friction for pairs of materials are presented in Table 3.1. Note that the ice and ice pair have very low coefficient of static friction. This means a block of ice can move easily over another block of ice.

Table 3.1 Coefficient of Static Friction for a Pair of Materials

Material	Coefficient of Static Friction
Glass and glass	1.0
Ice and ice	0.10
Steel and steel	0.75
Wood and wood	0.35
Rubber tyre and dry concrete road	1.0
Rubber tyre and wet road	0.7

3.6.3 Kinetic Friction

If the external force acting on the object is greater than maximum static friction, the objects begin to slide. When an object slides, the surface exerts a frictional force called **kinetic friction** $\vec{f}_k$ (also called sliding friction or dynamic friction). To move an object at constant velocity we must apply a force which is equal in magnitude and opposite to the direction of kinetic friction.

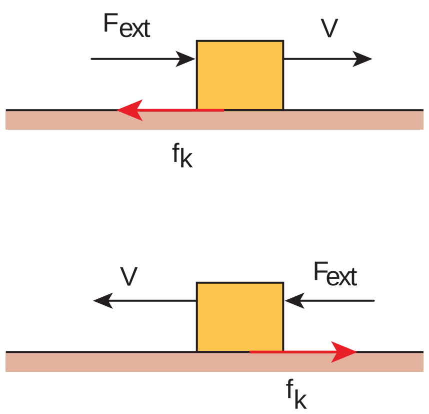


Figure 3.24 Kinetic friction

Experimentally it was found that the magnitude of kinetic friction satisfies the relation

$$f_k = \mu_k N \qquad (3.28)$$

where μ_k is the coefficient of kinetic friction and N the normal force exerted by the surface on the object,

and $\mu_k < \mu_s$

This implies that starting of a motion is more difficult than maintaining it. The salient features of static and kinetic friction are given in Table 3.2.

Table 3.2 Salient Features of Static and Kinetic Friction

Static friction	Kinetic friction
It opposes the starting of motion	It opposes the relative motion of the object with respect to the surface
Independent of surface area of contact	Independent of surface area of contact
μ_s depends on the nature of materials in mutual contact	μ_k depends on nature of materials and temperature of the surface
Depends on the magnitude of applied force	Independent of magnitude of applied force
It can take values from zero to $\mu_s N$	It can never be zero and always equals to $\mu_k N$ whatever be the speed (true v <10 ms^{-1})
$f_s^{max} > f_k$	It is less than maximum value of static friction
$\mu_s > \mu_k$	Coefficient of kinetic friction is less than coefficient of static friction

The variation of both static and kinetic frictional forces with external applied force is graphically shown in Figure 3.25.

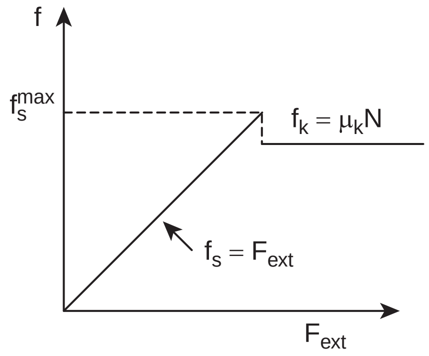


Figure 3.25 Variation of static and kinetic frictional forces with external applied force

The Figure 3.25 shows that static friction increases linearly with external applied force till it reaches the maximum. If the object begins to move then the kinetic friction is slightly lesser than the maximum static friction. Note that the kinetic friction is constant and it is independent of applied force.

Note The relation $f_s = \mu_s N$ is not a vector relation. This is because the normal force N and f_s are not in the same direction even though f_s is equal to μ_s times the normal force. This is also true in the case of kinetic friction.

3.6.4 To Move an Object - Push or pull? Which is easier?

When a body is pushed at an arbitrary angle θ $\left(0 \text{ to } \frac{\pi}{2}\right)$, the applied force F can be resolved into two components as F sinθ parallel to the surface and F cosθ perpendicular to the surface as shown in Figure 3.26. The total downward force acting on the body is mg + Fcosθ. It implies that the normal force acting on the body increases. Since there is no acceleration along the vertical direction the normal force N is equal to

$$N_{push} = mg + F cos\theta \qquad (3.29)$$

As a result the maximal static friction also increases and is equal to

$$f_s^{max} = \mu_s N_{push} = \mu_s\left(mg + F\, cos\theta\right) \qquad (3.30)$$

Equation (3.30) shows that a greater force needs to be applied to push the object into motion.

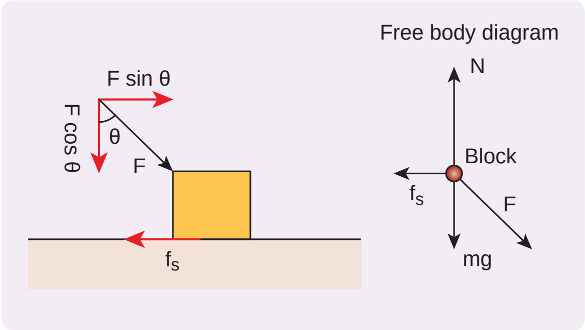


Figure 3.26 An object is pushed at an angle θ

When an object is pulled at an angle θ, the applied force is resolved into two components as shown in Figure 3.27. The total downward force acting on the object is

$$N_{pull} = mg - F cos\theta \qquad (3.31)$$

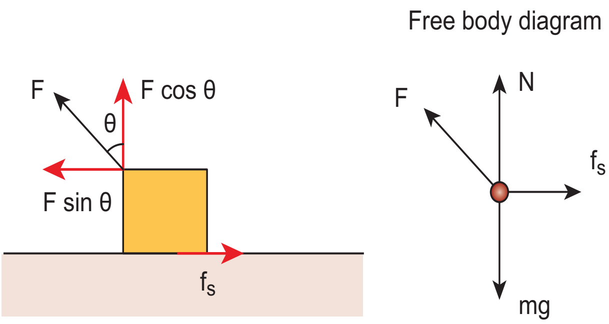


Figure 3.27 An object is pulled at an angle θ

Equation (3.31) shows that the normal force is less than N_{push}. From equations (3.29) and (3.31), it is easier to pull an object than to push to make it move.

3.6.5 Angle of Friction

The angle of friction is defined as the angle between the normal force (N) and the resultant force (R) of normal force and maximum friction force (f_s^{max})

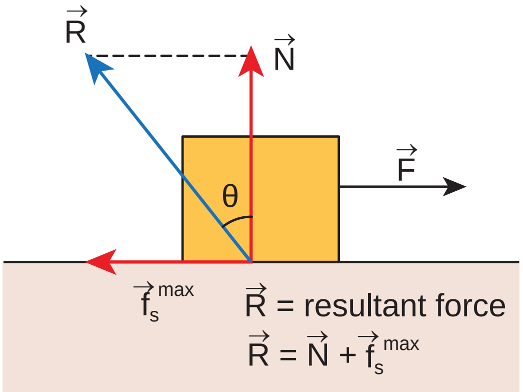


Figure 3.28 Angle of Friction

In Figure 3.28 the resultant force is $R = \sqrt{\left(f_s^{max}\right)^2 + N^2}$

$$\tan\theta = \frac{f_s^{max}}{N} \qquad (3.32)$$

But from the frictional relation, the object begins to slide when $f_s^{max} = \mu_s N$

$$\text{or when } \frac{f_s^{max}}{N} = \mu_s \qquad (3.33)$$

From equations (3.32) and (3.33) the coefficient of static friction is

$$\mu_s = tan\theta \qquad (3.34)$$

The coefficient of static friction is equal to tangent of the angle of friction

GBV47

3.6.6 Angle of Repose

Consider an inclined plane on which an object is placed, as shown in Figure 3.29. Let the angle which this plane makes with the horizontal be θ. For small angles of θ, the object may not slide down. As θ is increased, for a particular value of θ, the object begins to slide down. This value is called angle of repose. Hence, the angle of repose is the angle of inclined plane with the horizontal such that an object placed on it begins to slide.

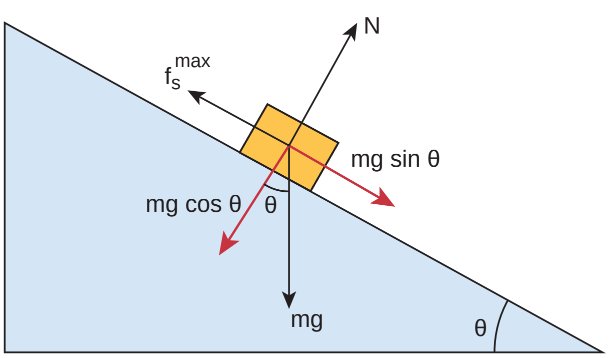


Figure 3.29 Angle of repose

Let us consider the various forces in action here. The gravitational force mg is resolved into components parallel ($mg\sin\theta$) and perpendicular ($mg\cos\theta$) to the inclined plane.

The component of force parallel to the inclined plane (mg sinθ) tries to move the object down.

The component of force perpendicular to the inclined plane (mg cosθ) is balanced by the Normal force (N).

$$N = mg\cos\theta$$

When the object just begins to move, the static friction attains its maximum value

$$f_s = f_s^{max} = \mu_s N = \mu_s mg\cos\theta \qquad (3.35)$$

This friction also satisfies the relation

$$f_s^{max} = mg\sin\theta \qquad (3.36)$$

Dividing equations (3.35) and (3.36), we get

$$\mu_s = \sin\theta / \cos\theta$$

From the definition of angle of friction, we also know that

$$\tan\theta = \mu_s, \qquad (3.37)$$

in which θ is the angle of friction.

Thus the angle of repose is the same as angle of friction. But the difference is that the angle of repose refers to inclined surfaces and the angle of friction is applicable to any type of surface.

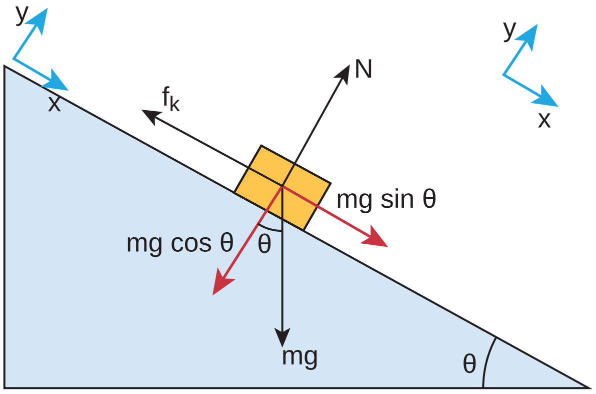
EXAMPLE 3.20

A block of mass m slides down the plane inclined at an angle 60° with an acceleration $\frac{g}{2}$. Find the coefficient of kinetic friction?

Solution

Kinetic friction comes to play as the block is moving on the surface.

The forces acting on the mass are the normal force perpendicular to surface, downward gravitational force and kinetic friction f_k along the surface.



Along the x-direction

$$mg\sin\theta - f_k = \text{ma}$$

But a = g/2

$$mg\sin 60° - f_k = \text{mg}/2$$

$$\frac{\sqrt{3}}{2}\text{mg} - f_k = \text{mg}/2$$

$$f_k = mg\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$f_K = \left(\frac{\sqrt{3}-1}{2}\right)\text{mg}$$

There is no motion along the y-direction as normal force is exactly balanced by the mg $\cos\theta$.

$$\text{mg}\cos\theta = \text{N} = \text{mg}/2$$

$$f_K = \mu_K \text{N} = \mu_K \text{mg}/2$$

$$\mu_K = \frac{\left(\frac{\sqrt{3}-1}{2}\right)mg}{\frac{mg}{2}}$$

$$\mu_K = \sqrt{3} - 1$$

3.6.7 Application of Angle of Repose

1. Antlions make sand traps in such a way that when an insect enters the edge of the trap, it starts to slide towards the bottom where the antilon hide itself. The angle of inclination of sand trap is made to be equal to angle of repose. It is shown in the Figure 3.30.

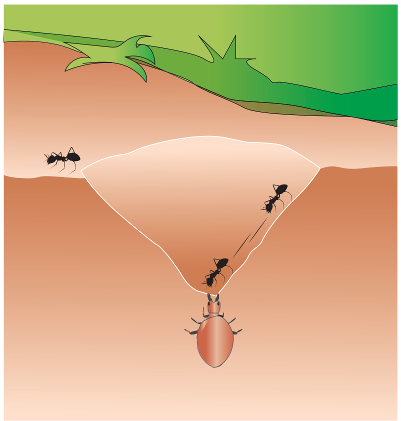


Figure 3.30 Sand trap of antlions

2. Children are fond of playing on sliding board (Figure 3.31). Sliding will be easier

UNIT-3(XI-Physics_Vol-1).indd 143 05-01-2022 22:18:24

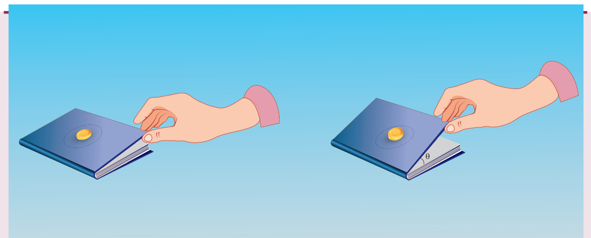
when the angle of inclination of the board is greater than the angle of repose. At the same time if inclination angle is much larger than the angle of repose, the slider will reach the bottom at greater speed and get hurt.

Figure 3.31 Sliding board

ACTIVITY

Measuring the coefficients of friction

Take a hard bound note book and a coin. Keep the coin on the note book. The note book cover has to be in an inclined position as shown in the figure. Slowly increase the angle of inclination of the cover with respect to rest of the pages. When the angle of inclination reaches the angle of repose, the parallel component of gravitational force (mg sinθ) to book surface becomes equal to the frictional force and the coin begins to slide down. Measure the angle of inclination and take the tangent of this angle. It gives the coefficient of static friction between the surface of the cover and coin. The same can be repeated with other objects such as an eraser in



order to observe that the coefficient of static friction differs from case to case.

Note At the point of sliding $\tan\theta_s = \mu_s$

To measure the coefficient of kinetic friction, reduce the inclination of the book after it starts sliding, such that the coin/eraser moves with uniform velocity. Now measure the angle from which coefficient of kinetic friction can be calculated as

$$\mu_K = \tan\theta_k$$

Observe that $\theta_K < \theta_S$

3.6.8 Rolling Friction

The invention of the wheel plays a crucial role in human civilization. One of the important applications is suitcases with rolling on coasters. Rolling wheels makes it easier than carrying luggage. When an object moves on a surface, essentially it is sliding on it. But wheels move on the surface through rolling motion. In rolling motion when a wheel moves on a surface, the point of contact with surface is always at rest. Since the point of contact is at rest, there is no relative motion between the wheel and surface. Hence the frictional force is very less. At the same time if an object moves without a wheel, there is a relative motion between the object and the surface. As a result frictional force is larger. This makes it difficult to move the object. The Figure 3.32 shows the difference between rolling and kinetic friction.

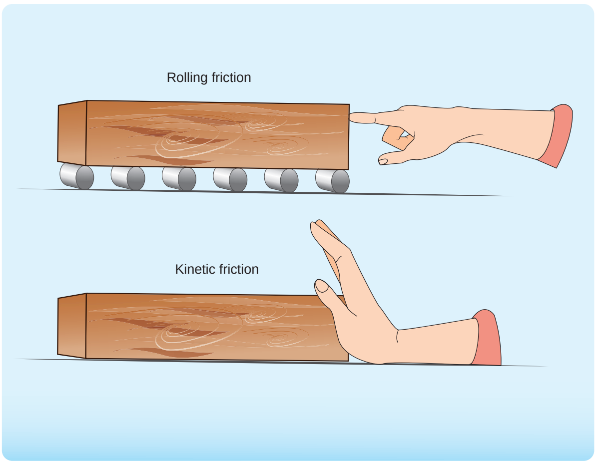


Figure 3.32 Rolling and kinetic friction

Ideally in pure rolling, motion of the point of contact with the surface should be at rest, but in practice it is not so. Due to the elastic nature of the surface at the point of contact there will be some deformation on the object at this point on the wheel or surface as shown in Figure 3.33. Due to this deformation, there will be minimal friction between wheel and surface. It is called 'rolling friction'. In fact, 'rolling friction' is much smaller than kinetic friction.

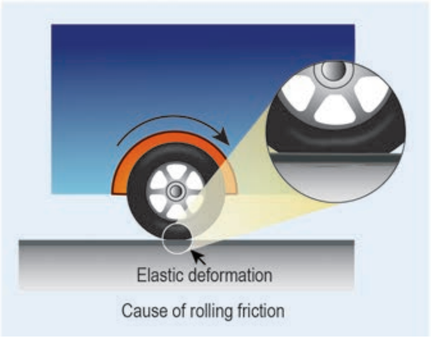


Figure 3.33 Rolling friction

3.6.9 Methods to Reduce Friction

Frictional force has both positive and negative effects. In some cases it is absolutely necessary. Walking is possible because of frictional force. Vehicles (bicycle, car) can move because of the frictional force between the tyre and the road. In the braking system, kinetic friction plays a major role. As we have already seen, the frictional force comes into effect whenever there is relative motion between two surfaces. In big machines used in industries, relative motion between different parts of the machine produce unwanted heat which reduces its efficiency. To reduce this kinetic friction lubricants are used as shown in Figure 3.34.

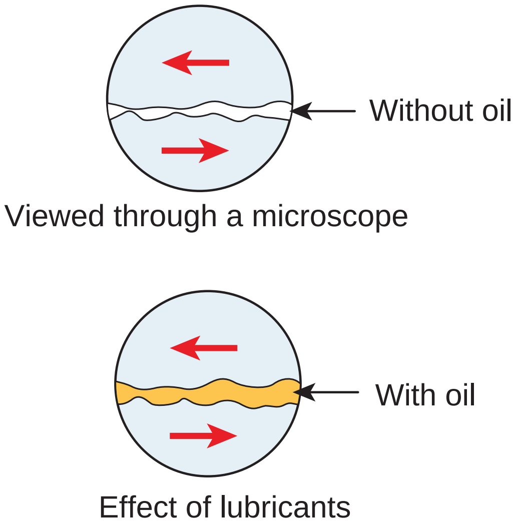


Figure 3.34 Reducing kinetic friction using lubricant

Ball bearings provides another effective way to reduce the kinetic friction (Figure 3.35) in machines. If ball bearings are fixed between two surfaces, during the relative motion only the rolling friction comes to effect and not kinetic friction. As we have seen earlier, the rolling friction is much smaller than kinetic friction; hence the machines are protected from wear and tear over the years.

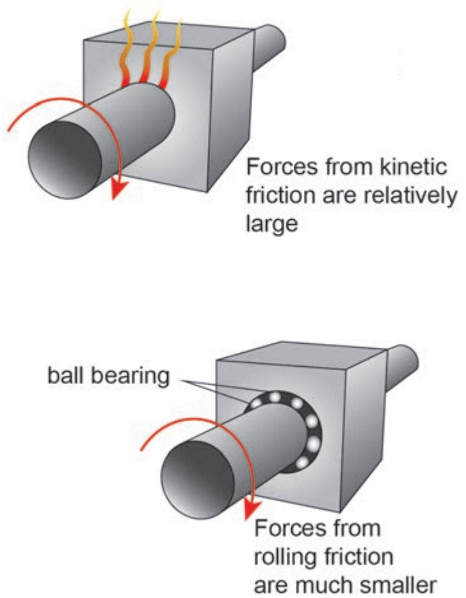


Figure 3.35 Reducing kinetic friction using ball bearing

During the time of Newton and Galileo, frictional force was considered as one of the natural forces like gravitational force. But in the twentieth century, the understanding on atoms, electron and protons has changed the perspective. The frictional force is actually the electromagnetic force between the atoms on the two surfaces. Even well polished surfaces have irregularities on the surface at the microscopic level as seen in the Figure 3.36.

POINTS TO PONDER

When you walk on the tiled floor where water is spilled, you are likely to slip. Why?

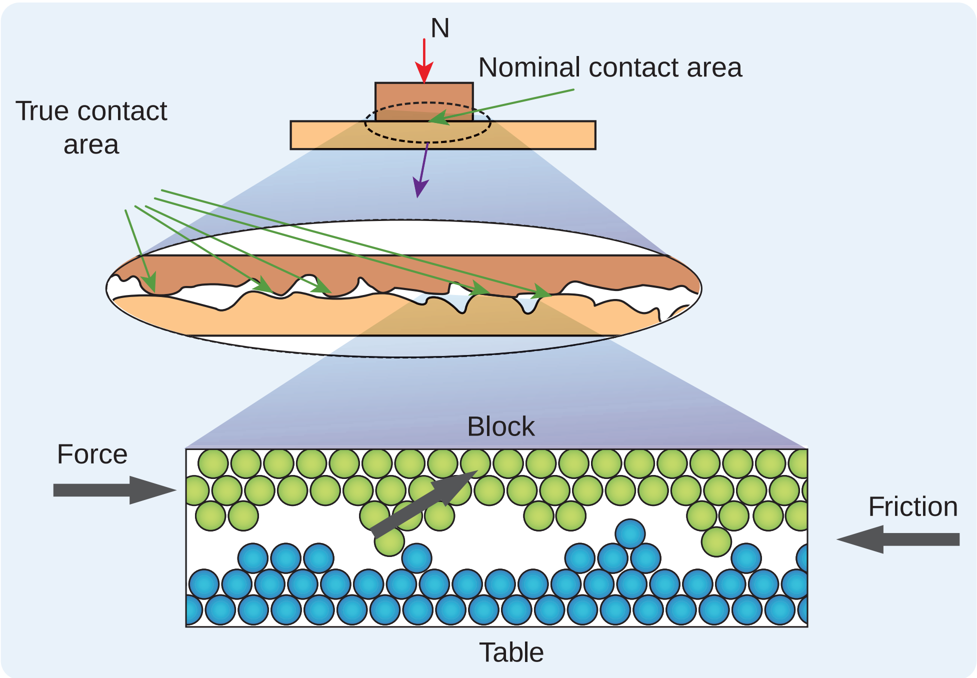
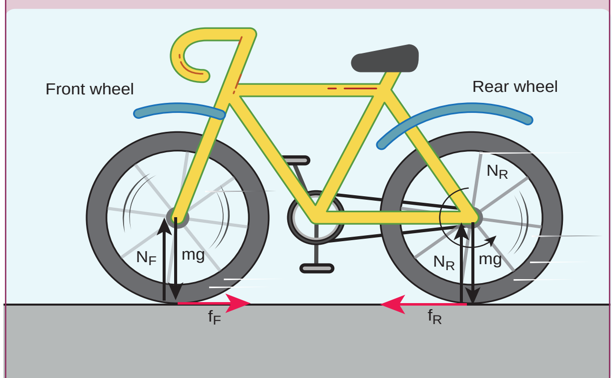


Figure 3.36 Irregularities on the surface at the microscopic level

Frictional force in the motion of a bicycle

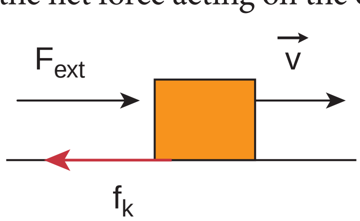
When a bicycle moves in the forward direction, what is the direction of frictional force in the rear and front wheels?



When we pedal a bicycle, we try to push the surface backward and the velocity of point of contact in the rear wheel is backwards. So, the frictional force pushes the rear wheel to move forward. But as the front wheel is connected with a rigid support to the back wheel, the forward motion of back wheel pushes the front wheel in the forward direction. So, the frictional forces act backward. Remember both frictional forces correspond to only static friction and not kinetic friction. If the wheel slips then kinetic friction comes into effect. In addition to static friction, the rolling friction also acts on both wheels in the backward direction.

EXAMPLE 3.21

Consider an object moving on a horizontal surface with a constant velocity. Some external force is applied on the object to keep the object moving with a constant velocity. What is the net force acting on the object?



Solution

If an object moves with constant velocity, then it has no acceleration. According to Newton's second law there is no net force acting on the object. The external force is balanced by the kinetic friction.

Note It is not that 'no force acts on the object'. In fact there are two forces acting on the object. Only the net force acting on the object is zero.

3.7 DYNAMICS OF CIRCULAR MOTION

In the previous sections we have studied how to analyse linear motion using Newton's laws. It is also important to know how to apply Newton's laws to circular motion, since circular motion is one of the very common types of motion that we come across in our daily life. A particle can be in linear motion with or without any external force. But when circular motion occurs there must necessarily be some force acting on the object. There is no Newton's first law for circular motion. In other words without a force, circular motion cannot occur in nature. A force can change the velocity of a particle in three different ways.

1. The magnitude of the velocity can be changed without changing the direction of the velocity. In this case the particle will move in the same direction but with acceleration.

Examples

Particle falling down vertically, bike moving in a straight road with acceleration.

2. The direction of motion alone can be changed without changing the magnitude (speed). If this happens continuously then we call it 'uniform circular motion'.
3. Both the direction and magnitude (speed) of velocity can be changed. If this happens non circular motion occurs. For example oscillation of a swing or simple pendulum, elliptical motion of planets around the Sun.

In this section we will deal with uniform circular motion and non-uniform circular motion.

3.7.1 Centripetal force

If a particle is in uniform circular motion, there must be centripetal acceleration towards the centre of the circle. If there is acceleration then there must be some force acting on it with respect to an inertial frame. This force is called centripetal force.

As we have seen in chapter 2, the centripetal acceleration of a particle in the circular motion is given by $a = \frac{v^2}{r}$ and it acts towards centre of the circle. According to Newton's second law, the centripetal force is given by

$$F_{cp} = m a_{cp} = \frac{mv^2}{r}$$

The word Centripetal force means centre seeking force.

In vector notation $\vec{F}_{cp} = -\frac{mv^2}{r}\hat{r}$

For uniform circular motion $\vec{F}_{cp} = -m\omega^2 r\,\hat{r}$

The direction $-\hat{r}$ points towards the centre of the circle which is the direction of centripetal force as shown in Figure 3.38.

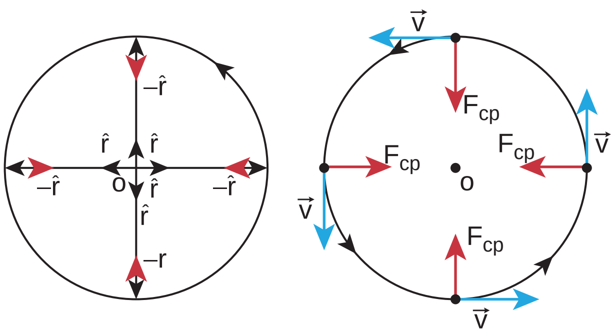


Figure 3.38 Centripetal force

It should be noted that 'centripetal force' is not other forces like gravitational force or spring force. It can be said as 'force towards centre'. The origin of the centripetal force can be gravitational force, tension in the string, frictional force, Coulomb force etc. Any of these forces can act as a centripetal force.

1. In the case of whirling motion of a stone tied to a string, the centripetal force on the particle is provided by the tensional force on the string. In circular motion in an amusement park, the centripetal force is provided by the tension in the iron ropes.
2. In motion of satellites around the Earth, the centripetal force is given by Earth's gravitational force on the satellites. Newton's second law for satellite motion is

$$F = \text{earth's gravitational force} = \frac{mv^2}{r}$$

Where r - distance of the planet from the centre of the Earth.

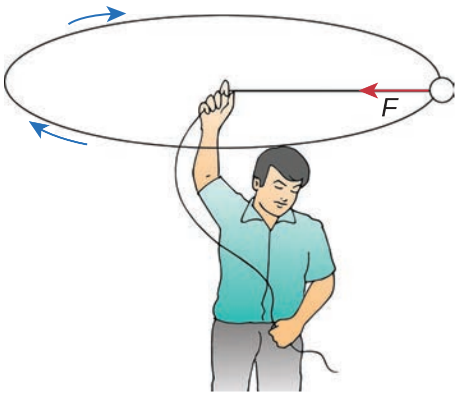


Figure 3.39 Whirling motion of objects

m-mass of the satellite

v-speed of the satellite

3. When a car is moving on a circular track the centripetal force is given by the frictional force between the road and the tyres.

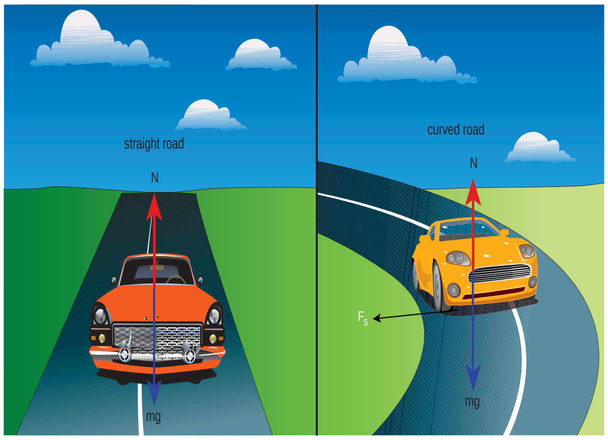


Figure 3.40 Car in the circular track

Newton's second law for this case is

$$\text{Frictional force} = \frac{mv^2}{r}$$

m-mass of the car

v-speed of the car

r-radius of curvature of track

Even when the car moves on a curved track, the car experiences the centripetal force which is provided by frictional force between the surface and the tyre of the car. This is shown in the Figure 3.41.

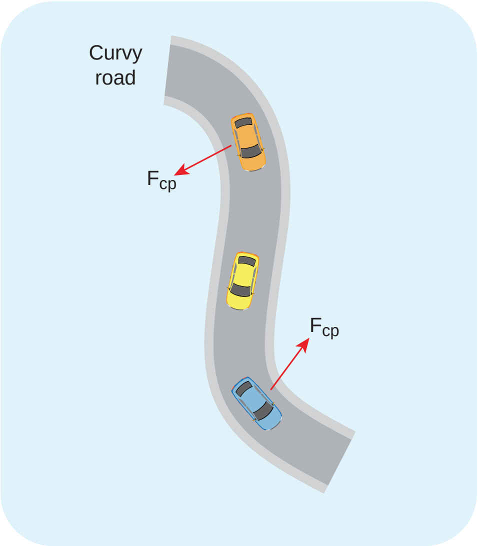


Figure 3.41 Centripetal force due to frictional force between the road and tyre

4. When the planets orbit around the Sun, they experience centripetal force towards the centre of the Sun. Here gravitational force of the Sun acts as centripetal force on the planets as shown in Figure 3.42

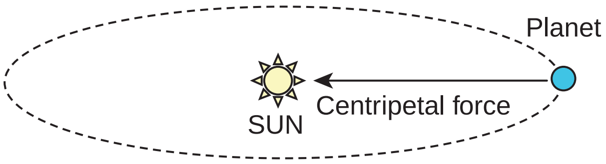


Figure 3.42 Centripetal force on the orbiting planet due Sun's gravity

Newton's second law for this motion

Gravitational force of Sun on the planet $= \frac{mv^2}{r}$

EXAMPLE 3.22

If a stone of mass 0.25 kg tied to a string executes uniform circular motion with a speed of 2 m s^{-1} of radius 3 m, what is the magnitude of tensional force acting on the stone?

Solution: $F_{cp} = \frac{mv^2}{r}$

$$F_{cp} = \frac{\frac{1}{4}\times(2)^2}{3} = 0.333 \text{ N.}$$

EXAMPLE 3.23

The Moon orbits the Earth once in 27.3 days in an almost circular orbit. Calculate the centripetal acceleration experienced by the Moon? (Radius of the Earth is 6.4×10^6 m)

Solution

The centripetal acceleration is given by $a = \frac{v^2}{r}$. This expression explicitly depends on Moon's speed which is non trivial. We can work with the formula

$$\omega^2 R_m = a_m$$

a_m is centripetal acceleration of the Moon due to Earth's gravity.

ω is angular velocity.

R_m is the distance between Earth and the Moon, which is 60 times the radius of the Earth.

$$R_m = 60R = 60 \times 6.4 \times 10^6 = 384 \times 10^6\ m$$

As we know the angular velocity $\omega = \frac{2\pi}{T}$ and T = 27.3 days = 27.3 × 24 × 60 × 60 second = 2.358×10^6 sec

By substituting these values in the formula for acceleration

$$a_m = \frac{(4\pi^2)(384\times10^6)}{(2.358\times10^6)^2} = 0.00272 \text{ m s}^{-2}$$

The centripetal acceleration of Moon towards the Earth is 0.00272 m s^{-2}

Note

This result was calculated by Newton himself. In unit 6 we will use this result.

3.7.2 Vehicle on a leveled circular road

When a vehicle travels in a curved path, there must be a centripetal force acting on it. This centripetal force is provided by the frictional force between tyre and surface of the road. Consider a vehicle of mass 'm' moving at a speed 'v' in the circular track of radius 'r'. There are three forces acting on the vehicle when it moves as shown in the Figure 3.43

1. Gravitational force (mg) acting downwards
2. Normal force (N) acting upwards
3. Frictional force (F_s) acting horizontally inwards along the road

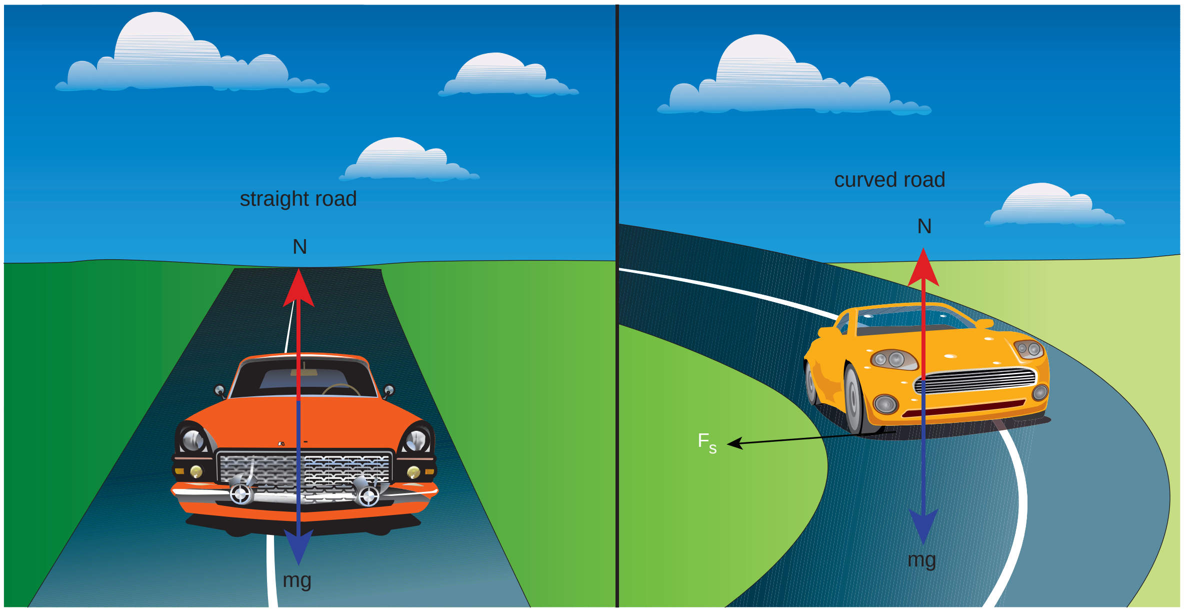


Figure 3.43 Forces acting on the vehicle on a leveled circular road

Suppose the road is horizontal then the normal force and gravitational force are exactly equal and opposite. The centripetal force is provided by the force of static friction F_s between the tyre and surface of the road which acts towards the centre of the circular track,

$$\frac{mv^2}{r} = F_s$$

As we have already seen in the previous section, the static friction can increase from zero to a maximum value

$$F_s \le \mu_s mg.$$

There are two conditions possible:

a) If $\frac{mv^2}{r} \le \mu_s mg$, or $\mu_s \ge \frac{v^2}{rg}$ or $\sqrt{\mu_s rg} \ge v$ (Safe turn)

The static friction would be able to provide necessary centripetal force to bend the car on the road. So the coefficient of static friction between the tyre and the surface of the road determines what maximum speed the car can have for safe turn.

b) If $\frac{mv^2}{r} > \mu_s mg$, or $\mu_s < \frac{v^2}{rg}$ (skid)

If the static friction is not able to provide enough centripetal force to turn, the vehicle will start to skid.

EXAMPLE 3.24

Consider a circular leveled road of radius 10 m having coefficient of static friction 0.81. Three cars (A, B and C) are travelling with speed 7 m s^{-1}, 8 m s^{-1} and 10 ms^{-1} respectively. Which car will skid when it moves in the circular level road? ($g = 10$ m s^{-2})

Solution

From the safe turn condition the speed of the vehicle (v) must be less than or equal to $\sqrt{\mu_s rg}$

$$v \le \sqrt{\mu_s rg}$$

$$\sqrt{\mu_s rg} = \sqrt{0.81 \times 10 \times 10} = 9\,m\,s^{-1}$$

For Car C, $\sqrt{\mu_s rg}$ is less than v

The speed of car A, B and C are 7 m s^{-1}, 8 m s^{-1} and 10 m s^{-1} respectively. The cars A and B will have safe turns. But the car C has speed 10 m s^{-1} while it turns which exceeds the safe turning speed. Hence, the car C will skid.

3.7.3 Banking of Tracks

In a leveled circular road, skidding mainly depends on the coefficient of static friction μ_s The coefficient of static friction depends on the nature of the surface which has a maximum limiting value. To avoid this problem, usually the outer edge of the road is slightly raised compared to inner edge as shown in the Figure 3.44. This is called banking of roads or tracks. This introduces an inclination, and the angle is called banking angle.

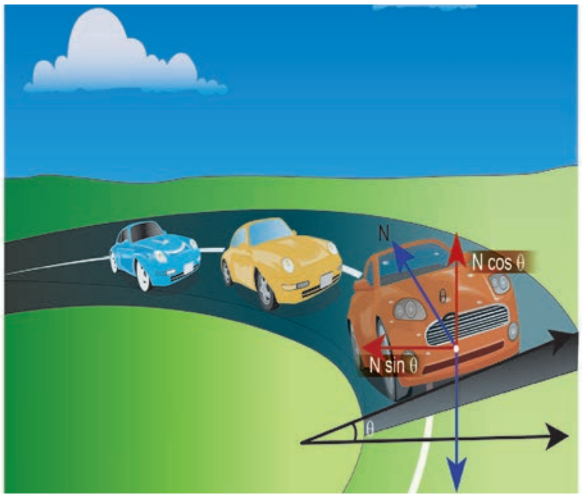


Figure 3.44 Outer edge of the road is slightly raised to avoid skidding

Let the surface of the road make angle θ with horizontal surface. Then the normal force makes the same angle θ with the vertical. When the car takes a turn, there are two forces acting on the car:

a) Gravitational force mg (downwards)

b) Normal force N (perpendicular to surface)

We can resolve the normal force into two components. $N\cos\theta$ and $N\sin\theta$ as shown in Figure 3.46. The component $N\cos\theta$ balances the downward gravitational force 'mg' and component $N\sin\theta$ will provide the necessary centripetal acceleration. By using Newton second law

$$N\cos\theta = mg$$

$$N\sin\theta = \frac{mv^2}{r}$$

By dividing the equations we get $\tan\theta = \frac{v^2}{rg}$

$$v = \sqrt{rg\tan\theta}$$

The banking angle θ and radius of curvature of the road or track determines the safe speed of the car at the turning. If the speed of car exceeds this safe speed, then it starts to skid outward but frictional force comes into effect and provides an additional centripetal force to prevent the outward skidding. At the same time, if the speed of the car is little lesser than safe speed, it starts to skid inward and frictional force comes into effect, which reduces centripetal force to prevent inward skidding. However if the speed of the vehicle is sufficiently greater than the correct speed, then frictional force cannot stop the car from skidding.

EXAMPLE 3.25

Consider a circular road of radius 20 meter banked at an angle of 15 degree. With what speed a car has to move on the turn so that it will have safe turn?

Solution

$$v = \sqrt{(rg\tan\theta)} = \sqrt{20 \times 9.8 \times \tan 15°}$$

$$= \sqrt{20 \times 9.8 \times 0.26} = 7.1 \text{ m s}^{-1}$$

The safe speed for the car on this road is 7.1 m s^{-1}

3.7.4 Centrifugal Force

Circular motion can be analysed from two different frames of reference. One is the inertial frame (which is either at rest or in uniform motion) where Newton's laws are obeyed. The other is the rotating frame of reference which is a non-inertial frame of reference as it is accelerating. When we examine the circular motion from these frames of reference the situations are entirely different. To use Newton's first and second laws in the rotational frame of reference, we need to include a pseudo force called 'centrifugal force'. This 'centrifugal force' appears to act on the object with respect to rotating frames. To understand the concept of centrifugal force, we can take a specific case and discuss as done below.

Consider the case of a whirling motion of a stone tied to a string. Assume that the stone has angular velocity ω in the inertial frame (at rest). If the motion of the stone is observed from a frame which is also rotating along with the stone with same angular velocity ω then, the stone appears to be at rest. This implies that in addition to the inward centripetal force $-m\omega^2 r$ there must be an equal and opposite force that acts on the stone outward with value $+m\omega^2 r$. So the total force acting on the stone in a rotating frame is equal to zero $(-m\omega^2 r + m\omega^2 r = 0)$. This outward force $+m\omega^2 r$ is called the centrifugal force. The word 'centrifugal' means 'flee from centre'. Note that the 'centrifugal force' appears to act on the particle, only when we analyse the motion from a rotating frame. With respect to an inertial frame there is only centripetal force which is given by the tension in the string. For this reason centrifugal force is called as a 'pseudo force'. A pseudo force has no origin. It arises due to the non inertial nature of the frame considered. When circular motion problems are solved from a rotating frame of reference, while drawing free body diagram of a particle, the centrifugal force should necessarily be included as shown in the Figure 3.45.

3.7.5 Effects of Centrifugal Force

Although centrifugal force is a pseudo force, its effects are real. When a car takes a turn in a curved road, person inside the car feels an outward force which pushes the person away. This outward force is also called centrifugal force. If there is sufficient friction between the person and the seat, it will prevent the person from moving outwards. When a car moving in a straight line suddenly takes a turn, the objects not fixed to the car try to continue in linear motion due to their inertia of direction. While observing this motion from an inertial frame, it appears as a straight line as shown in Figure 3.46. But, when it is observed from the rotating frame it appears to move outwards.

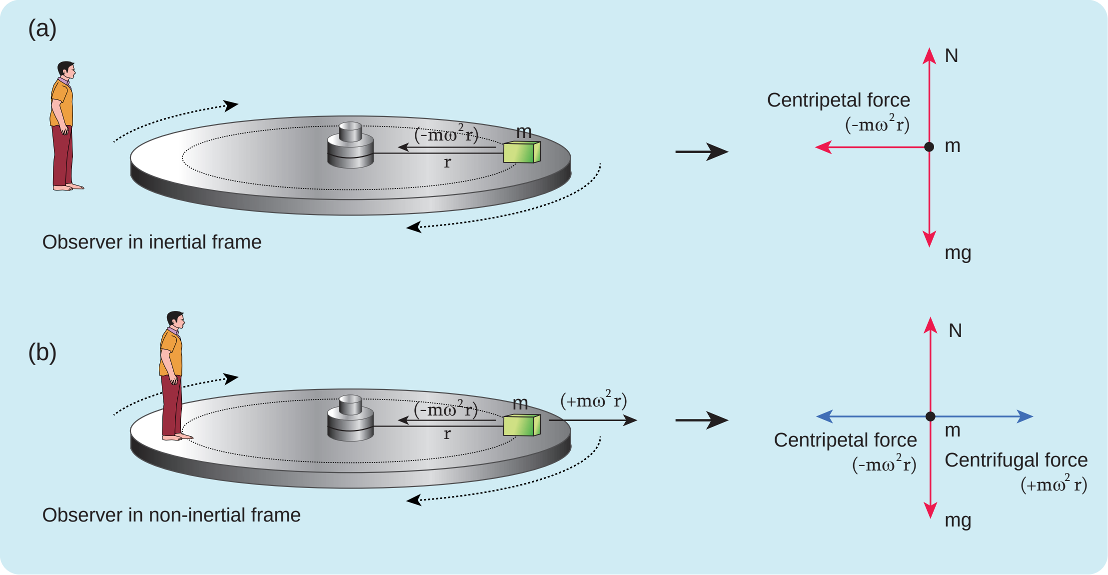


Figure 3.45 Free body diagram of a particle including the centrifugal force

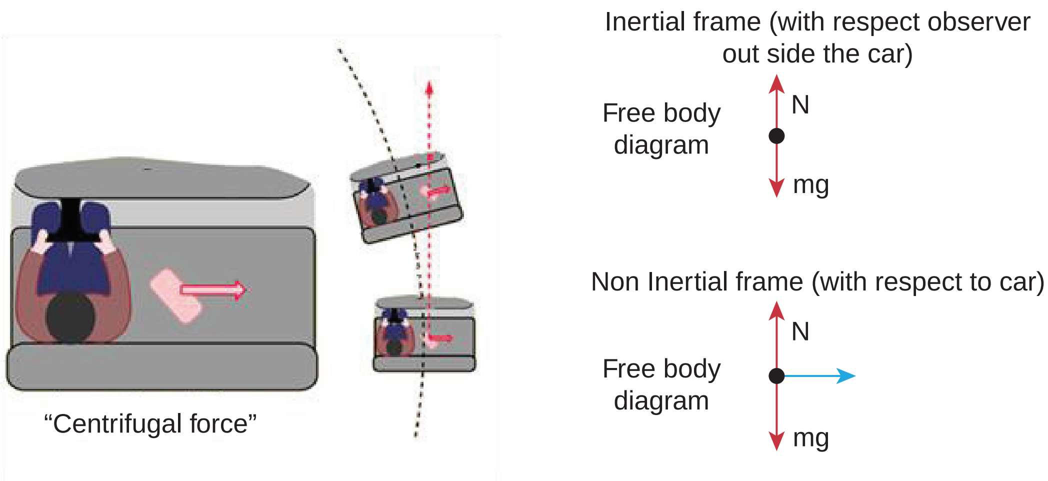


Figure 3.46 Effects of centrifugal force

A person standing on a rotating platform feels an outward centrifugal force and is likely to be pushed away from the platform. Many a time the frictional force between the platform and the person is not sufficient to overcome outward push. To avoid this, usually the outer edge of the platform is little inclined upwards which exerts a normal force on the person which prevents the person from falling as illustrated in Figures 3.47.

Caution!

It is dangerous to stand near the open door (or) steps while travelling in the bus. When the bus takes a sudden turn in a curved road, due to centrifugal force the person is pushed away from the bus. Even though centrifugal force is a pseudo force, its effects are real.

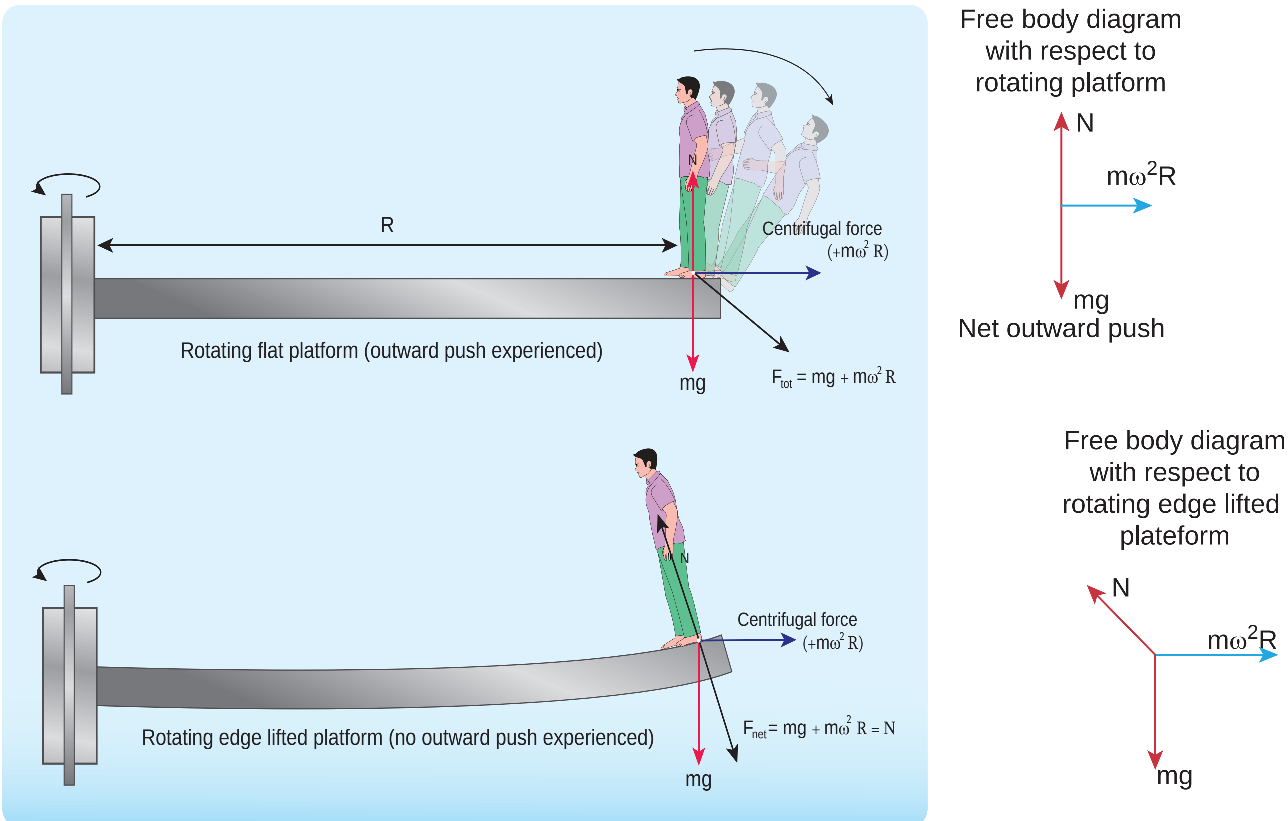


Figure 3.47 Outward centrifugal force in rotating platform

3.7.6 Centrifugal Force due to Rotation of the Earth

Even though Earth is treated as an inertial frame, it is actually not so. Earth spins about its own axis with an angular velocity ω. Any object on the surface of Earth (rotational frame) experiences a centrifugal force. The centrifugal force appears to act exactly in opposite direction from the axis of rotation. It is shown in the Figure 3.48.

The centrifugal force on a man standing on the surface of the Earth is $F_{cf} = m\omega^2 r$

where r is perpendicular distance of the man from the axis of rotation. By using right angle triangle as shown in the Figure 3.48, the distance $r = R\cos\theta$

Here R = radius of the Earth

and θ = latitude of the Earth where the man is standing.

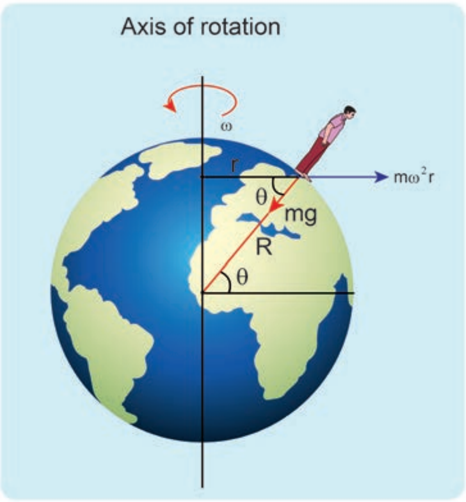


Figure 3. 48 Centrifugal force acting on a man on the surface of Earth

EXAMPLE 3.26

Calculate the centrifugal force experienced by a man of 60 kg standing at Chennai? (Given: Latitude of Chennai is $13°$

Solution

The centrifugal force is given by $F_c = m\omega^2 R\cos\theta$

The angular velocity (ω) of Earth $= \frac{2\pi}{T}$, where T is time period of the Earth (24 hours)

$$\omega = \frac{2\pi}{24\times60\times60} = \frac{2\pi}{86400}$$
$$= 7.268\times10^{-5}\,rad sec^{-1}$$

The radius of the Earth R = 6400 Km = 6400×10^3 m

Latitude of Chennai $= 13°$

$$F_{cf} = 60\times\left(7.268\times10^{-5}\right)^2\times6400\times10^3$$
$$\times\cos\left(13°\right) = 1.9678\,N$$

A 60 kg man experiences centrifugal force of approximately 2 Newton. But due to Earth's gravity a man of 60 kg experiences a force = mg = 60 × 9.8 = 588N. This force is very much larger than the centrifugal force.

3.7.7 Centripetal Force Versus Centrifugal Force

Salient features of centripetal and centrifugal forces are compared in Table 3.4.

Table 3.4 Salient Features of Centripetal and Centrifugal Forces

Centripetal force	Centrifugal force
It is a real force which is exerted on the body by the external agencies like gravitational force, tension in the string, normal force etc.	It is a pseudo force or fictitious force which cannot arise from gravitational force, tension force, normal force etc.
Acts in both inertial and non-inertial frames	Acts only in rotating frames (non-inertial frame)
It acts towards the axis of rotation or centre of the circle in circular motion $\left\|F_{cp}\right\| = m\omega^2 r = \frac{mv^2}{r}$	It acts outwards from the axis of rotation or radially outwards from the centre of the circular motion $\left\|F_{cf}\right\| = m\omega^2 r = \frac{mv^2}{r}$
Real force and has real effects	Pseudo force but has real effects
Origin of centripetal force is interaction between two objects.	Origin of centrifugal force is inertia. It does not arise from interaction. In an inertial frame the object's inertial motion appears as centrifugal force in the rotating frame.
In inertial frames centripetal force has to be included when free body diagrams are drawn.	In inertial frames there is no centrifugal force. In rotating frames, both centripetal and centrifugal force have to be included when free body diagrams are drawn.

SUMMARY

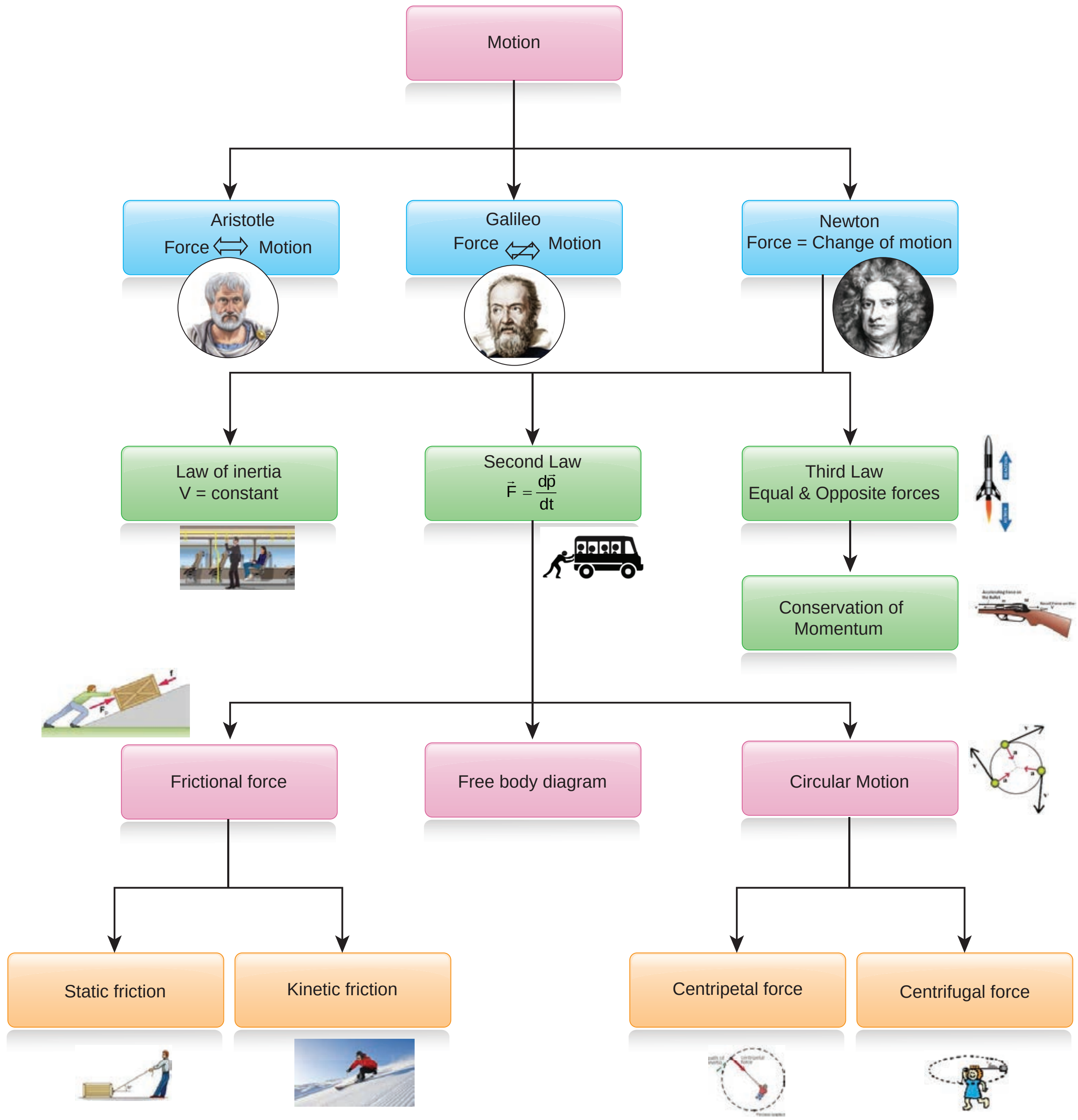
- Aristotle's idea of motion: To maintain motion, a force is required
- Galileo's idea of motion: To maintain motion, a force is not required
- Mass is a measure of inertia of the body
- Newton's first law states that under no external force, the object continues its state of motion or state of rest.
- Newton's second law states that to change the momentum of the body, external force is required

  Mathematically it is defined as $\vec{F} = \frac{d\vec{p}}{dt}$
- Both Newton's first and second laws are valid only in inertial frames
- Inertial frame is the one in which if there is no force on the object, the object moves at constant velocity.
- Newton's third law states that for every force there is an equivalent and opposite force and such a pair of forces is called action and reaction pair.
- To draw a free body diagram for an object,
  - Isolate the object from other objects and identify the forces acting on it
  - The force exerted by that object should not be taken into account
  - Draw the direction of each force with relative magnitude
  - Apply Newton's second law in each direction
- If no net external force acts on a collection of particles (system), then the total momentum of the collection of particles (system) is a constant vector.
- Internal forces acting in the system cannot change the total momentum of the system.
- Lami's theorem states that if an object is in equilibrium under the concurrent forces, then the ratio of each force with the sine of corresponding opposite angle is same.
- An impulse acting on a body is equal to the change in momentum of the body. Whenever a force acts on the object for a very short time, it is difficult to calculate the force. But impulse can be calculated.
- Static friction is the force which always opposes the movement of the object from rest. It can take values from zero to $\mu_s N$. If an external force is greater than $\mu_s N$ then object begins to move.
- If the object begins to move, kinetic friction comes into effect. To move an object with constant velocity, the external force must be applied to overcome the kinetic friction. The kinetic friction is $\mu_k N$.
- Rolling friction is much smaller than static and kinetic friction. This is the reason that to move an object roller coaster is fixed in the bottom of the object. Example: Rolling suitcase

SUMMARY (*cont*)

- The origin of friction is electromagnetic interaction between the atoms of two surfaces which are touching each other.
- Whenever there is a motion along a curve, there must be a centripetal force that acts towards the centre of the curve. In uniform circular motion the centripetal force acts at the centre of the circle.
- The centripetal force is not a separate natural force. Any natural force can behave as centripetal force. In planetary motion, Sun's gravitational force acts as centripetal force. In the whirling motion of a stone attached to a string, the centripetal force is given by the string. When Moon orbits the Earth, it experiences Earth's gravitational force as centripetal force.
- Centrifugal force arises whenever the motion is analysed from rotating frame. It is a pseudo force. The inertial motion of the object appears as centrifugal force in the rotating frame.
- The magnitude of centrifugal and centripetal force is $m\omega^2 r$. But centripetal force acts towards centre of the circular motion and centrifugal force appears to acts in the opposite direction to centripetal force.

CONCEPT MAP



Motion
Aristotle
Force ⟺ Motion
Galileo
Force ⇎ Motion
Newton
Force = Change of motion
Law of inertia
V = constant
Second Law
$\vec{F} = \frac{d\vec{p}}{dt}$
Third Law
Equal & Opposite forces
Conservation of Momentum
Frictional force
Free body diagram
Circular Motion
Static friction
Kinetic friction
Centripetal force
Centrifugal force

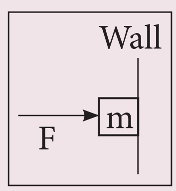
EVALUATION

I. Multiple Choice Questions

1. When a car takes a sudden left turn in the curved road, passengers are pushed towards the right due to
   (a) inertia of direction
   (b) inertia of motion
   (c) inertia of rest
   (d) absence of inertia

2. An object of mass m held against a vertical wall by applying horizontal force F as shown in the figure. The minimum value of the force F is

   (IIT JEE 1994)

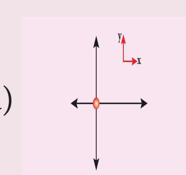
   (a) Less than mg
   (b) Equal to mg
   (c) Greater than mg
   (d) Cannot determine

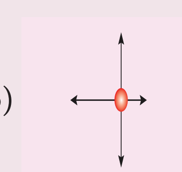
3. A vehicle is moving along the positive x direction, if sudden brake is applied, then
   (a) frictional force acting on the vehicle is along negative x direction
   (b) frictional force acting on the vehicle is along positive x direction
   (c) no frictional force acts on the vehicle
   (d) frictional force acts in downward direction

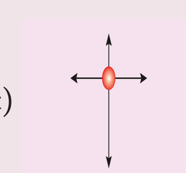
4. A book is at rest on the table which exerts a normal force on the book. If this force is considered as reaction force, what is the action force according to Newton's third law?
   (a) Gravitational force exerted by Earth on the book
   (b) Gravitational force exerted by the book on Earth
   (c) Normal force exerted by the book on the table
   (d) None of the above

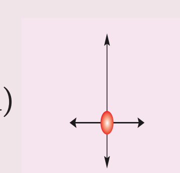
5. Two masses m_1 and m_2 are experiencing the same force where $m_1 < m_2$. The ratio of their acceleration $\frac{a_1}{a_2}$ is
   (a) 1
   (b) less than 1
   (c) greater than 1
   (d) all the three cases

6. Choose appropriate free body diagram for the particle experiencing net acceleration along negative y direction. (Each arrow mark represents the force acting on the system).

   a) 

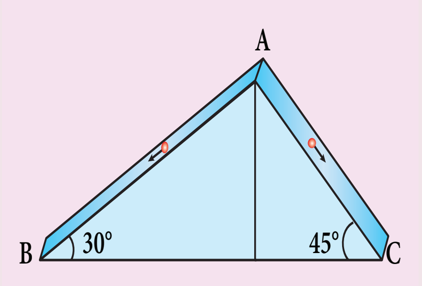
   b) 

   c) 

   d) 

7. A particle of mass m sliding on the smooth double inclined plane (shown in figure) will experience

UNIT-3(XI-Physics_Vol-1).indd 160 05-01-2022 22:18:46



(a) greater acceleration along the path AB

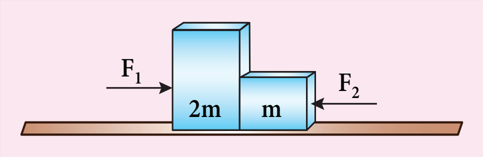
(b) greater acceleration along the path AC

(c) same acceleration in both the paths

(d) no acceleration in both the paths.

8. Two blocks of masses m and 2m are placed on a smooth horizontal surface as shown. In the first case only a force F_1 is applied from the left. Later only a force F_2 is applied from the right. If the force acting at the interface of the two blocks in the two cases is same, then $F_1 : F_2$ is

(Physics Olympiad 2016)



(a) 1:1 (b) 1:2

(c) 2:1 (d) 1:3

9. Force acting on the particle moving with constant speed is

(a) always zero

(b) need not be zero

(c) always non zero

(d) cannot be concluded

10. An object of mass m begins to move on the plane inclined at an angle θ. The coefficient of static friction of inclined surface is μ_s. The maximum static friction experienced by the mass is

(a) mg

(b) $\mu_s$mg

(c) $\mu_s$mg sinθ

(d) μ_s mg cos θ

11. When the object is moving at constant velocity on the rough surface,

(a) net force on the object is zero

(b) no force acts on the object

(c) only external force acts on the object

(d) only kinetic friction acts on the object

12. When an object is at rest on the inclined rough surface,

(a) static and kinetic frictions acting on the object is zero

(b) static friction is zero but kinetic friction is not zero

(c) static friction is not zero and kinetic friction is zero

(d) static and kinetic frictions are not zero

13. The centrifugal force appears to exist

(a) only in inertial frames

(b) only in rotating frames

(c) in any accelerated frame

(d) both in inertial and non-inertial frames

14. Choose the correct statement from the following
    (a) Centrifugal and centripetal forces are action reaction pairs
    (b) Centripetal forces is a natural force
    (c) Centrifugal force arises from gravitational force
    (d) Centripetal force acts towards the centre and centrifugal force appears to act away from the centre in a circular motion

15. If a person moving from pole to equator, the centrifugal force acting on him
    (a) increases
    (b) decreases
    (c) remains the same
    (d) increases and then decreases

Answers

1) a	**2)** c	**3)** a	**4)** c	**5)** c
6) c	**7)** b	**8)** c	**9)** b	**10)** d
11) a	**12)** c	**13)** b	**14)** d	**15)** a

II. Short Answer Questions

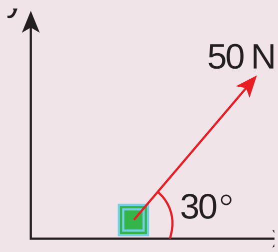
1. Explain the concept of inertia. Write two examples each for inertia of motion, inertia of rest and inertia of direction.
2. State Newton's second law.
3. Define one newton.
4. Show that impulse is the change of momentum.
5. Using free body diagram, show that it is easy to pull an object than to push it.
6. Explain various types of friction. Suggest a few methods to reduce friction.
7. What is the meaning by 'pseudo force'?
8. State the empirical laws of static and kinetic friction.
9. State Newton's third law.
10. What are inertial frames?
11. Under what condition will a car skid on a leveled circular road?

III. Long Answer Questions

1. Prove the law of conservation of linear momentum. Use it to find the recoil velocity of a gun when a bullet is fired from it.
2. What are concurrent forces? State Lami's theorem.
3. Explain the motion of blocks connected by a string in i) Vertical motion ii) Horizontal motion.
4. Briefly explain the origin of friction. Show that in an inclined plane, angle of friction is equal to angle of repose.
5. State Newton's three laws and discuss their significance.
6. Explain the similarities and differences of centripetal and centrifugal forces.
7. Briefly explain 'centrifugal force' with suitable examples.
8. Briefly explain 'rolling friction'.
9. Describe the method of measuring angle of repose.
10. Explain the need for banking of tracks.
11. Calculate the centripetal acceleration of Moon towards the Earth.

IV. Numerical Problems

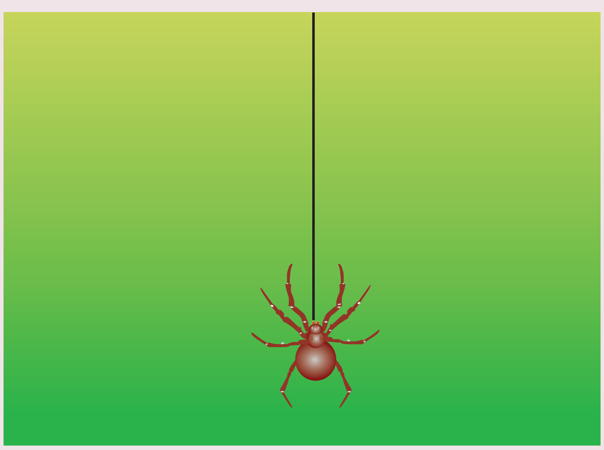
1. A force of 50N act on the object of mass 20 kg. shown in the figure. Calculate the acceleration of the object in x and y directions.



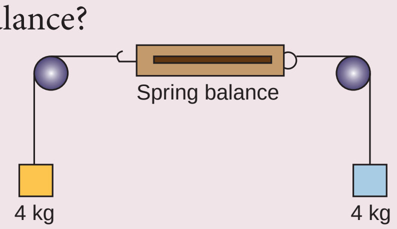
Ans: a_x=2.165 ms^{-2}; a_y=1.25 ms^{-2}

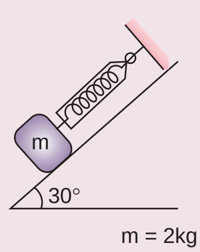
2. A spider of mass 50 g is hanging on a string of a cob web as shown in the figure. What is the tension in the string?

Ans: T=0.49N



3. What is the reading shown in spring balance?





Ans: Zero, 9.8 N

4. The physics books are stacked on each other in the sequence: +1 volumes 1 and 2; +2 volumes 1 and 2 on a table.

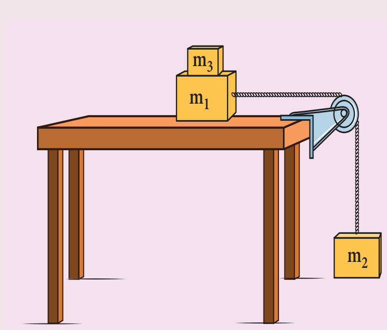
a) Identify the forces acting on each book and draw the free body diagram.

b) Identify the forces exerted by each book on the other.

5. A bob attached to the string oscillates back and forth. Resolve the forces acting on the bob in to components. What is the acceleration experience by the bob at an angle θ.

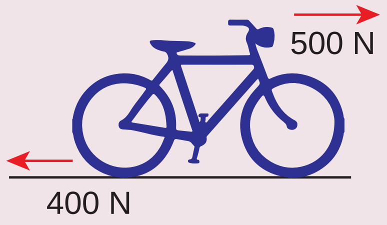
Ans: Tangential acceleration = g sinθ ; centripetal acceleration = $\dfrac{(T-mg\cos\theta)}{m}$.

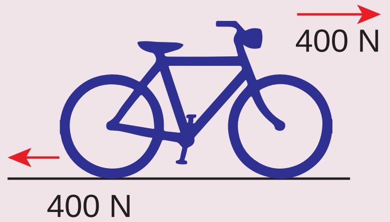
6. Two masses m_1 and m_2 are connected with a string passing over a frictionless pulley fixed at the corner of the table as shown in the figure. The coefficient of static friction of mass m_1 with the table is μ_s. Calculate the minimum mass m_3 that may be placed on m1to prevent it from sliding. Check if m_1=15 kg, m_2=10 kg, m_3=25 kg and μ_s =0.2



Ans: $m_3 = \dfrac{m_2}{\mu_s} - m_1$, the combined masses m_1+m_3 will slide.

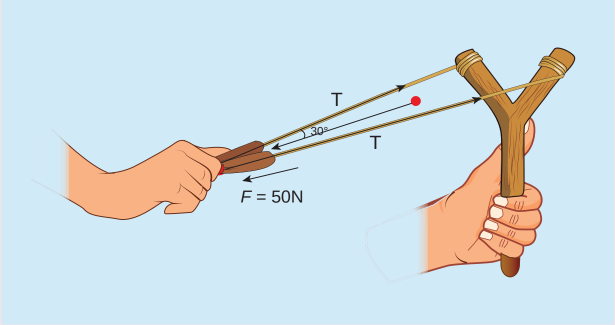
7. Calculate the acceleration of the bicycle of mass 25 kg as shown in Figures 1 and 2.





Ans: a=4 ms^{-2}, zero

8. Apply Lami's theorem on sling shot and calculate the tension in each string ?



Ans: $T= 28.868N$.

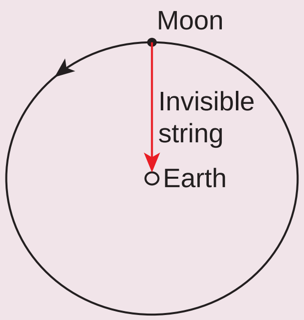
9. A football player kicks a 0.8 kg ball and imparts it a velocity 12 ms^{-1}. The contact between the foot and ball is only for one- sixtieth of a second. Find the average kicking force.

Ans: $576N$.

10. A stone of mass 2 kg is attached to a string of length 1 meter. The string can withstand maximum tension 200 N. What is the maximum speed that stone can have during the whirling motion?

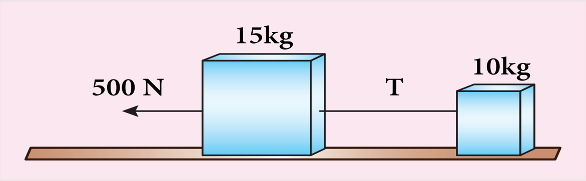
Ans: $v_{max}=10ms^{-1}$

11. Imagine that the gravitational force between Earth and Moon is provided by an invisible string that exists between the Moon and Earth. What is the tension that exists in this invisible string due to Earth's centripetal force? (Mass of the Moon = 7.34×10^{22} kg, Distance between Moon and Earth = 3.84×10^{8} m)



Ans: $T \approx 2 \times 10^{20} N$.

12. Two bodies of masses 15 kg and 10 kg are connected with light string kept on a smooth surface. A horizontal force F=500 N is applied to a 15 kg as shown in the figure. Calculate the tension acting in the string



Ans: $T = 200N$.

13. People often say "For every action there is an equivalent opposite reaction". Here they meant 'action of a human'. Is it correct to apply Newton's third law to human actions? What is mean by 'action' in Newton third law? Give your arguments based on Newton's laws.

Ans: Newton's third law is applicable to only human's actions which involves physical force. Third law is not applicable to human's psychological actions or thoughts

14. A car takes a turn with velocity 50 ms^{-1} on the circular road of radius of curvature 10 m. calculate the centrifugal force experienced by a person of mass 60kg inside the car?

Ans: 15,000 N

15. A long stick rests on the surface. A person standing 10 m away from the stick. With what minimum speed an object of mass 0.5 kg should he thrown so that it hits the stick. (Assume the coefficient of kinetic friction is 0.7).

Ans: 11.71 ms^{-1}

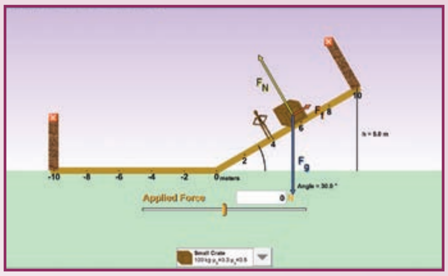
BOOKS FOR REFERENCE

1. Charles Kittel, Walter Knight, Malvin Ruderman, Carl Helmholtz and Moyer, *Mechanics,* 2nd edition, Mc Graw Hill Pvt Ltd,
2. A.P.French, *Newtonian Mechanics,* Viva-Norton Student edition
3. SomnathDatta, *Mechanics,* Pearson Publication
4. H.C.Verma, *Concepts of physics* volume 1 and Volume 2, Bharati Bhawan Publishers
5. Serway and Jewett, *Physics for scientist and Engineers with modern physics*, Brook/Coole publishers, Eighth edition
6. Halliday, Resnick & Walker, *Fundamentals of Physics*, Wiley Publishers, 10th edition

ICT CORNER

Force and motion

Through this activity you will understand the Force and motion

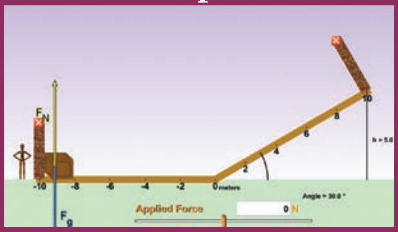


STEPS:

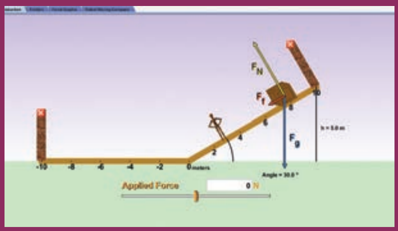
- Open the browser and type the given URL to open the PhET simulation on force and motion. Click OK to open the java applet.
- Select the values of the applied force to observe the change.
- Observe the change of the ramp angle by changing the position of the object.
- You can also observe the variations in force and ramp angle by changing the weights.

Step1

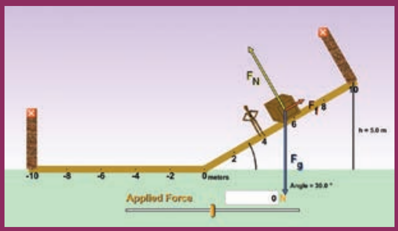
Step2



Step3



Step4



PhET simulation's URL:

https://phet.colorado.edu/en/simulation/ramp-forces-and-motion

* Pictures are indicative only.

* If browser requires, allow **Flash Player** or **Java Script** to load the page.

UNIT 4

WORK, ENERGY AND POWER

"Matter is Energy. Energy is Light. We are all Light Beings." – Albert Einstein

LEARNING OBJECTIVES

In this unit, student is exposed to

- definition of work
- work done by a constant and a variable force
- various types of energy
- law of conservation of energy
- vertical circular motion
- definition of power
- various types of collisions

8GPTA

4.1 INTRODUCTION

The term *work* is used in diverse contexts in daily life. It refers to both physical as well as mental work. In fact, any activity can generally be called as work. But in Physics, the term work is treated as a physical quantity with a precise definition. Work is said to be done by the force when the force applied on a body displaces it. To do work, *energy* is required. In simple words, *energy is defined as the ability to do work*. Hence, work and energy are equivalents and have same dimension. Energy, in Physics exists in different forms such as mechanical, electrical, thermal, nuclear and so on. Many machines consume one form of energy and deliver energy in a different form. In this chapter we deal mainly with mechanical energy and its two types namely kinetic energy and potential energy. The next quantity in this sequence of discussion is the rate of work done or the rate of energy delivered. *The rate of work done is called power*. A powerful strike in cricket refers to a hit on the ball at a fast rate. This chapter aims at developing a good understanding of these three physical quantities namely work, energy and power and their physical significance.

4.1.1 WORK

Let us consider a force ($\vec{F}$), acting on a body which moves it by a displacement in some direction ($d\vec{r}$) as shown in Figure 4.1

The expression for work done (W) by the force on the body is mathematically written as,

$$W = \vec{F} \cdot d\vec{r} \qquad (4.1)$$

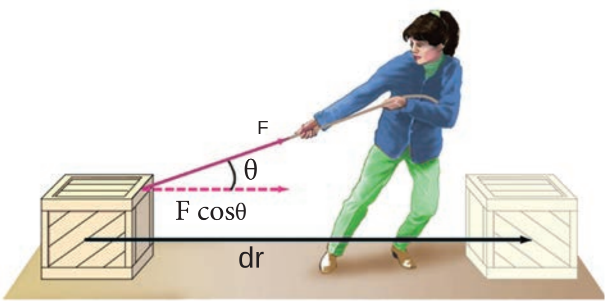


Figure 4.1 Work done by a force

Here, the product $\vec{F} \cdot d\vec{r}$ is a *scalar product (or dot product)*. The scalar product of two vectors is a scalar (refer section 2.5.1). Thus, work done is a scalar quantity. It has only magnitude and no direction. In SI system, unit of work done is N m (or) joule (J). Its dimensional formula is $[ML^2T^{-2}]$.

The equation (4.1) is,

$$W = F\, dr \cos\theta \tag{4.2}$$

which can be realised using Figure 4.2 (as $\vec{a} \cdot \vec{b} = ab\cos\theta$) where, θ is the angle between applied force and the displacement of the body.

The work done by the force depends on the force (F), displacement (dr) and the angle (θ) between them.

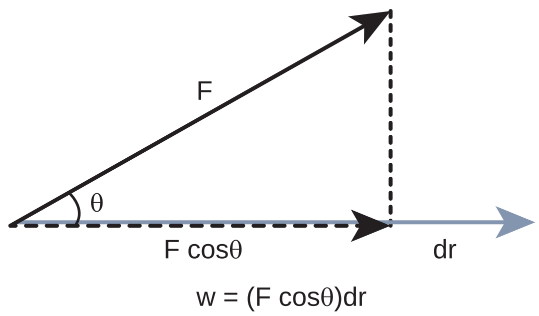


Figure 4.2 Calculating work done.

Work done is zero in the following cases.

(i) **When the force is zero (F = 0).** For example, a body moving on a horizontal smooth frictionless surface will continue to do so as no force (not even friction) is acting along the plane. (This is an ideal situation.)

(ii) **When the displacement is zero (dr = 0).** For example, when force is applied on a rigid wall it does not produce any displacement. Hence, the work done is zero as shown in Figure. 4.3(a).

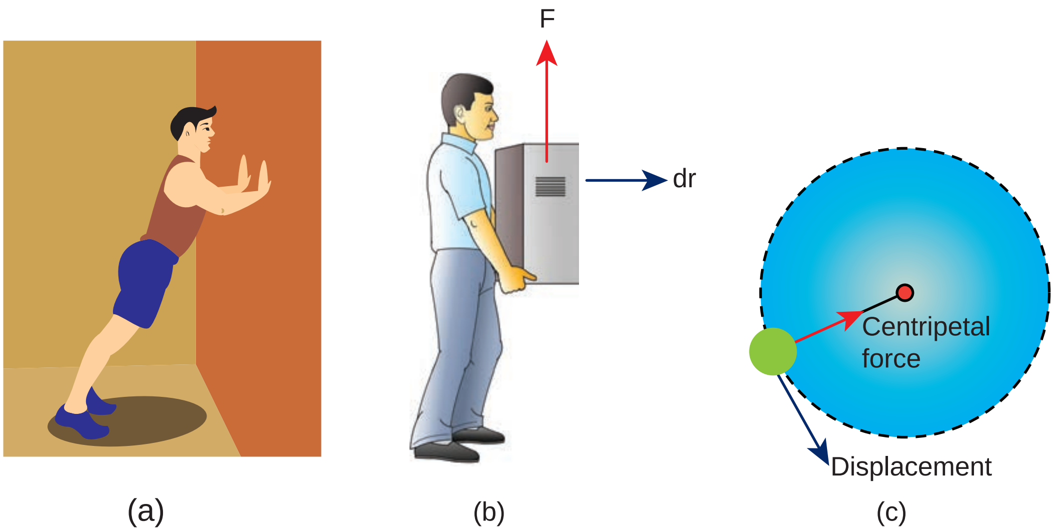


Figure 4.3 Different cases of zero work done

(iii) **When the force and displacement are perpendicular ($\theta = 90^o$) to each other.** when a body moves on a horizontal direction, the gravitational force (mg) does no work on the body, since it acts at right angles to the displacement as shown in Figure 4.3(b). In circular motion the centripetal force does not do work on the object moving on a circle as it is always perpendicular to the displacement as shown in Figure 4.3(c).

For a given force (F) and displacement (dr), the angle (θ) between them decides the value of work done as consolidated in Table 4.1.

There are many examples for the negative work done by a force. In a football game, the goalkeeper catches the ball coming towards him by applying a force such that the force is applied in a direction opposite to that of the motion of the ball till it comes to rest in his hands. During the time of applying the force, he does a negative work on the ball as shown in Figure 4.4. We will discuss many more situations of negative work further in this unit.

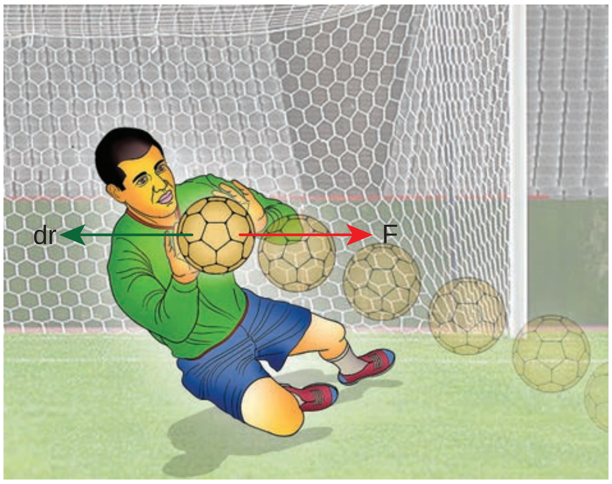
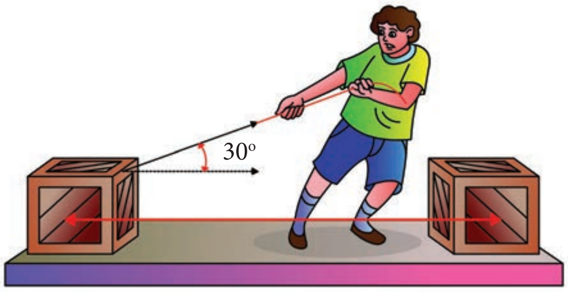


Figure 4.4 Negative work done

EXAMPLE 4.1

A box is pulled with a force of 25 N to produce a displacement of 15 m. If the angle between the force and displacement is 30^o, find the work done by the force.



Solution

Force, F = 25 N
Displacement, dr = 15 m
Angle between F and dr, $\theta = 30^o$

Table 4.1 Angle (θ) and the nature of work

Angle (θ)	cosθ	Work
$\theta = 0^o$	1	Positive, Maximum
$0 < \theta < 90^o$ (acute)	$0 < \cos\theta < 1$	Positive
$\theta = 90^o$ (right angle)	0	Zero
$90^o < \theta < 180^o$	$-1 < \cos\theta < 0$	Negative
$\theta = 180^o$	-1	Negative, Maximum

Work done, $W = F\,dr\cos\theta$

$$W = 25\times15\times\cos30° = 25\times15\times\frac{\sqrt{3}}{2}$$

$$W = 324.76\ \text{J}$$

4.1.2 Work done by a constant force

When a constant force F acts on a body, the small work done (dW) by the force in producing a small displacement dr is given by the relation,

$$dW = (F\cos\theta)\,dr \qquad (4.3)$$

The total work done in producing a displacement from initial position $\mathbf{r}_i$ to final position $\mathbf{r}_f$ is,

$$W = \int_{r_i}^{r_f} dW \qquad (4.4)$$

$$W = \int_{r_i}^{r_f} (F\cos\theta)\,dr = (F\cos\theta)\int_{r_i}^{r_f} dr = (F\cos\theta)(r_f - r_i) \qquad (4.5)$$

The graphical representation of the work done by a constant force is shown in Figure 4.5. The area under the graph shows the work done by the constant force.

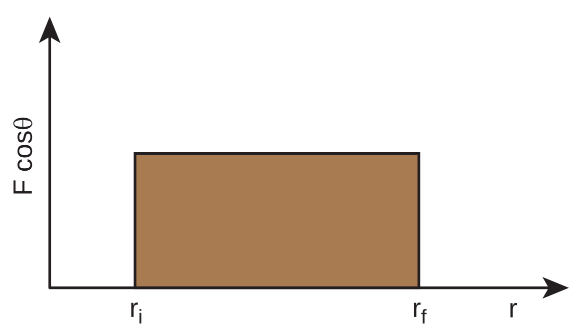
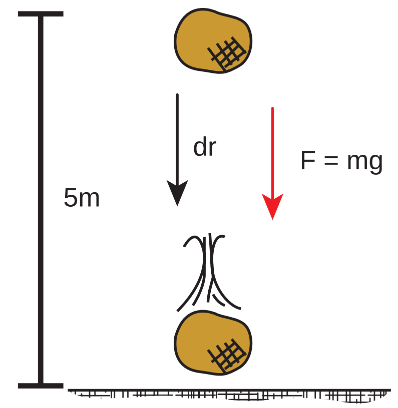


Figure 4.5 Work done by the constant force

EXAMPLE 4.2

An object of mass 2 kg falls from a height of 5 m to the ground. What is the work done by the gravitational force on the object? (Neglect air resistance; Take $g = 10\ \text{m s}^{-2}$)



Solution

In this case the force acting on the object is downward gravitational force $m\vec{g}$. This is a constant force.

Work done by gravitational force is

$$W = \int_{r_i}^{r_f} \vec{F}\cdot d\vec{r}$$

$$W = (F\cos\theta)\int_{r_i}^{r_f} dr = (mg\cos\theta)\,(r_f - r_i).$$

The object also moves downward which is in the direction of gravitational force ($\vec{F} = m\vec{g}$) as shown in figure. Hence, the angle between them is $\theta = 0°$; $\cos 0° = 1$ and the displacement, $(r_f - r_i) = 5\,\text{m}$

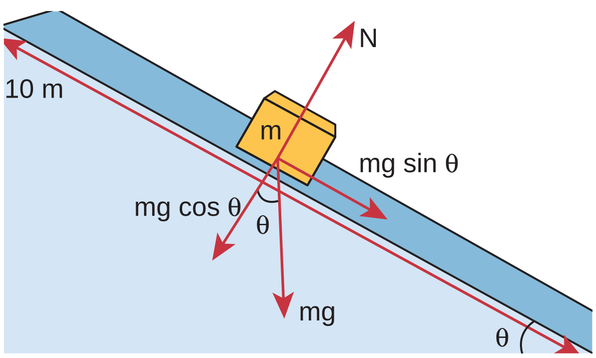
$$W = mg\,(r_f - r_i)$$

$$W = 2\times10\times5 = 100\ \text{J}$$

The work done by the gravitational force on the object is positive.

EXAMPLE 4.3

An object of mass m=1 kg is sliding from top to bottom in the frictionless inclined plane of inclination angle $\theta = 30°$ and the length of inclined plane is 10 m as shown in the figure. Calculate the work done by gravitational force and normal force on the object. Assume acceleration due to gravity, g = 10 m s^{-2}



Solution

We calculated in the previous chapter that the acceleration experienced by the object in the inclined plane as $g\sin\theta$.

According to Newton's second law, the force acting on the mass along the inclined plane $F = mg\sin\theta$. Note that this force is constant throughout the motion of the mass.

The work done by the parallel component of gravitational force $(mg\ \sin\theta)$ is given by

$$W = \vec{F}.d\vec{r} = Fdr\cos\phi$$

where ϕ is the angle between the force (mg sinθ) and the direction of motion (dr). In this case, force (mg sinθ) and the displacement ($d\vec{r}$) are in the same direction. Hence ϕ = 0 and cos ϕ = 1

$$W = F\ dr = (mg\ \sin\theta)\ (dr)$$

(dr = length of the inclined place)

$$W = 1 \times 10 \times \sin(30°) \times 10 = 100 \times \frac{1}{2} = 50\ J$$

The component mg cosθ and the normal force N are perpendicular to the direction of motion of the object, so they do not perform any work.

EXAMPLE 4.4

If an object of mass 2 kg is thrown up from the ground reaches a height of 5 m and falls back to the Earth (neglect the air resistance). Calculate

(a) The work done by gravity when the object reaches 5 m height

(b) The work done by gravity when the object comes back to Earth

(c) Total work done by gravity both in upward and downward motion and mention the physical significance of the result.

Solution

When the object goes up, the displacement points in the upward direction whereas the gravitational force acting on the object points in downward direction. Therefore, the angle between gravitational force and displacement of the object is 180°.

(a) The work done by gravitational force in the upward motion.

Given that $dr = 5m$ and $F = mg$

$$W_{up} = Fdr\ \cos\theta = mgdr\ \cos 180°$$

$$W_{up} = 2 \times 10 \times 5 \times (-1) = -100 \text{ joule.}$$

[cos 180° = −1]

(b) When the object falls back, both the gravitational force and displacement of the object are in the same direction. This implies that the angle between gravitational force and displacement of the object is 0°.

$$W_{down} = Fdr\cos 0°$$
$$W_{down} = 2\times10\times5\times(1) = 100 \text{ joule}$$
$$[\cos 0° = 1]$$

(c) The total work done by gravity in the entire trip (upward and downward motion)

$$W_{total} = W_{up} + W_{down}$$
$$= -100 \text{ joule} + 100 \text{ joule} = 0$$

It implies that the gravity does not transfer any energy to the object. When the object is thrown upwards, the energy is transferred to the object by the external agency, which means that the object gains some energy. As soon as it comes back and hits the Earth, the energy gained by the object is transferred to the surface of the Earth (i.e., dissipated to the Earth).

EXAMPLE 4.5

A weight lifter lifts a mass of 250 kg with a force 5000 N to the height of 5 m.

(a) What is the workdone by the weight lifter?
(b) What is the workdone by the gravity?
(c) What is the net workdone on the object?

Solution

(a) When the weight lifter lifts the mass, force and displacement are in the same direction, which means that the angle between them $\theta = 0^0$. Therefore, the work done by the weight lifter,

$$W_{weight\ lifter} = F_w h\ \cos\theta = F_w h\left(\cos 0^0\right)$$
$$= 5000\times5\times(1) = 25{,}000 \text{ joule} = 25 \text{ kJ}$$

(b) When the weight lifter lifts the mass, the gravity acts downwards which means that the force and displacement are in opposite direction. Therefore, the angle between them $\theta = 180^0$

$$W_{gravity} = F_g h\ \cos\theta = mgh\left(\cos 180^0\right)$$
$$= 250\times10\times5\times(-1)$$
$$= -12{,}500 \text{ joule} = -12.5 \text{ kJ}$$

(c) The net workdone (or total work done) on the object

$$W_{net} = W_{weight\ lifter} + W_{gravity}$$
$$= 25 \text{ kJ} - 12.5 \text{ kJ} = +12.5 \text{ kJ}$$

4.1.3 Work done by a variable force

When the component of a variable force F acts on a body, the small work done (dW) by the force in producing a small displacement dr is given by the relation

$$dW = (F\cos\theta)dr$$

[F cos θ is the component of the variable force F]

where, F and θ are variables. The total work done for a displacement from initial position r_i to final position r_f is given by the relation,

$$W = \int_{r_i}^{r_f} dW = \int_{r_i}^{r_f} F\cos\theta\ dr \qquad (4.6)$$

A graphical representation of the work done by a variable force is shown in Figure 4.6. The area under the graph is the work done by the variable force.

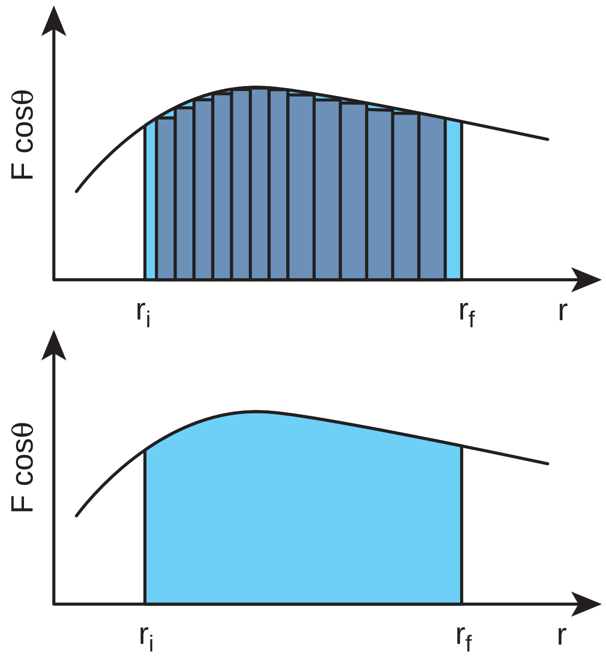


Figure 4.6 Work done by a variable force

EXAMPLE 4.6

A variable force $F = k\,x^2$ acts on a particle which is initially at rest. Calculate the work done by the force during the displacement of the particle from $x = 0$ m to $x = 4$ m. (Assume the constant $k = 1\ \mathrm{N\ m^{-2}}$)

Solution

Work done,

$$W = \int_{x_i}^{x_f} F(x)\,dx = k\int_0^4 x^2 dx = \frac{64}{3}\ \mathrm{N\ m}$$

4.2 ENERGY

Energy is defined as the capacity to do work. In other words, work done is the manifestation of energy. That is why work and energy have the same dimension (ML^2T^{-2})

Work ⇔ Energy

The important aspect of energy is that for an isolated system, the sum of all forms of energy i.e., the total energy remains the same in any process irrespective of whatever internal changes may take place. This means that the energy disappearing in one form reappears in another form. This is known as the law of conservation of energy. In this chapter we shall take up only the mechanical energy for discussion.

In a broader sense, mechanical energy is classified into two types

1. Kinetic energy
2. Potential energy

The energy possessed by a body due to its motion is called kinetic energy. The energy possessed by the body by virtue of its position is called potential energy.

The SI unit of energy is the same as that of work done i.e., N m (or) joule (J). The dimension of energy is also the same as that of work done. It is given by $[ML^2T^{-2}]$. The other units of energy and their SI equivalent values are given in Table 4.2.

Table 4.2 SI equivalent of other units of energy

Unit	Equivalent in joule
1 erg (CGS unit)	10^{-7} J
1 electron volt (eV)	$1.6\text{x}10^{-19}$ J
1 calorie (cal)	4.186 J
1 kilowatt hour (kWh)	$3.6\text{x}10^{6}$ J

4.2.1 Kinetic energy

Kinetic energy is the energy possessed by a body by virtue of its motion. All moving objects have kinetic energy. A body that is in motion has the ability to do work. For example a hammer kept at rest on a nail does not push the nail into the wood. Whereas the same hammer when it strikes the nail, draws the nail into the wood as shown in Figure 4.7. Kinetic energy is measured by the amount of work that the body can perform before it comes to rest. The amount of work done by a moving body depends both on the mass of the body and the magnitude of its velocity. A body which is not in motion does not have kinetic energy.

4.2.2 Work–Kinetic Energy Theorem

Work and energy are equivalents. This is true in the case of kinetic energy also. To prove this, let us consider a body of mass m at rest on a frictionless horizontal surface.

The work (W) done by the constant force (F) for a displacement (s) in the same direction is,

$$W = Fs \qquad (4.7)$$

The constant force is given by the equation,

$$F = ma \qquad (4.8)$$

The third equation of motion (refer section 2.10.3) can be written as,

$$v^2 = u^2 + 2as$$
$$a = \frac{v^2 - u^2}{2s}$$

Substituting for a in equation (4.8),

$$F = m\left(\frac{v^2 - u^2}{2s}\right) \qquad (4.9)$$

Substituting equation (4.9) in (4.7),

$$W = m\left(\frac{v^2}{2s}s\right) - m\left(\frac{u^2}{2s}s\right)$$
$$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \qquad (4.10)$$

The expression for kinetic energy:

The term $\left(\frac{1}{2}mv^2\right)$ in the above equation is the kinetic energy of the body of mass (m) moving with velocity (v).

$$KE = \frac{1}{2}mv^2 \qquad (4.11)$$

Kinetic energy of the body is always positive. From equations (4.10) and (4.11)

$$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \qquad (4.12)$$
Thus, $W = \Delta KE$

Figure 4.7 Demonstration of kinetic energy

The expression on the right hand side (RHS) of equation (4.12) is the change in kinetic energy (ΔKE) of the body.

This implies that *the work done by the force on the body changes the kinetic energy of the body. This is called work-kinetic energy theorem.*

The work-kinetic energy theorem implies the following.

1. If the work done by the force on the body is positive then its kinetic energy increases.
2. If the work done by the force on the body is negative then its kinetic energy decreases.
3. If there is no work done by the force on the body then there is no change in its kinetic energy, which means that the body has moved at constant speed provided its mass remains constant.

4.2.3 Relation between Momentum and Kinetic Energy

Consider an object of mass m moving with a velocity $\vec{v}$. Then its linear momentum is $\vec{p} = m\vec{v}$ and its kinetic energy, $KE = \frac{1}{2}mv^2$.

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\vec{v}.\vec{v}\right) \quad (4.13)$$

Multiplying both the numerator and denominator of equation (4.13) by mass, m

$$KE = \frac{1}{2}\frac{m^2\left(\vec{v}.\vec{v}\right)}{m}$$

$$= \frac{1}{2}\frac{\left(m\vec{v}\right).\left(m\vec{v}\right)}{m} \quad [\vec{p} = m\vec{v}]$$

$$= \frac{1}{2}\frac{\vec{p}.\vec{p}}{m}$$

$$= \frac{p^2}{2m}$$

$$KE = \frac{p^2}{2m} \quad (4.14)$$

where $|\vec{p}|$ is the magnitude of the momentum. The magnitude of the linear momentum can be obtained by

$$|\vec{p}| = p = \sqrt{2m\,(KE)} \quad (4.15)$$

Note that if kinetic energy and mass are given, only the magnitude of the momentum can be calculated but not the direction of momentum. It is because the kinetic energy and mass are scalars.

EXAMPLE 4.7

Two objects of masses 2 kg and 4 kg are moving with the same momentum of 20 kg m s^{-1}.

(a) Will they have same kinetic energy?
(b) Will they have same speed?

Solution

(a) The kinetic energy of the mass is given by $KE = \frac{p^2}{2m}$

For the object of mass 2 kg, kinetic energy is $KE_1 = \frac{(20)^2}{2\times 2} = \frac{400}{4} = 100\,J$

For the object of mass 4 kg, kinetic energy is $KE_2 = \frac{(20)^2}{2 \times 4} = \frac{400}{8} = 50\,J$

Note that $KE_1 \neq KE_2$ i.e., even though both are having the same momentum, the kinetic energy of both masses is not the same. The kinetic energy of the heavier object has lesser kinetic energy than smaller mass. It is because the kinetic energy is inversely proportional to the mass ($KE \propto \frac{1}{m}$) for a given momentum.

(b) As the momentum, $p = mv$, the two objects will not have same speed.

4.2.4 Potential Energy

The potential energy of a body is associated with its position and configuration with respect to its surroundings. This is because the various forces acting on the body also depends on position and configuration.

"Potential energy of an object at a point P is defined as the amount of work done by an external force in moving the object at constant velocity from the point O (initial location) to the point P (final location). At initial point O potential energy can be taken as zero.

Mathematically, *potential energy* is defined as $U = \int \vec{F}_a . d\vec{r}$ (4.16)

where the limit of integration ranges from initial location point O to final location point P.

We have various types of potential energies. Each type is associated with a particular force. For example,

(i) The energy possessed by the body due to gravitational force gives rise to gravitational potential energy.

(ii) The energy due to spring force and other similar forces give rise to elastic potential energy.

(iii) The energy due to electrostatic force on charges gives rise to electrostatic potential energy.

We will learn more about conservative forces in the section 4.2.7. Now, we continue to discuss more about gravitational potential energy and elastic potential energy.

4.2.5 Potential energy near the surface of the Earth

The gravitational potential energy (U) at some height *h* is equal to the amount of work required to take the object from ground to that height *h* with constant velocity.

Let us consider a body of mass *m* being moved from ground to the height h against the gravitational force as shown in Figure 4.8.

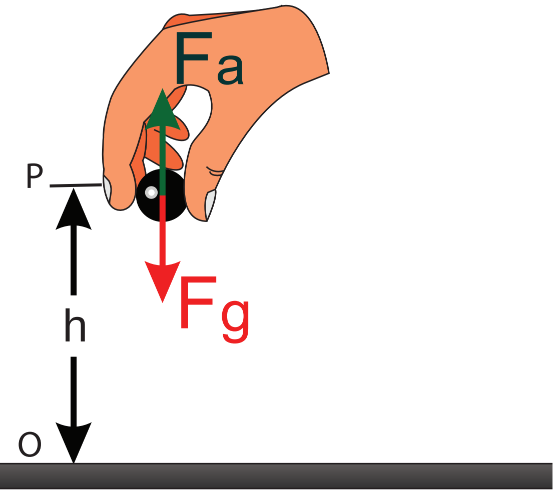


Figure 4.8 Gravitational potential energy

The gravitational force $\vec{F}_g$ acting on the body is, $\vec{F}_g = -mg\,\hat{j}$ (as the force is in y direction, unit vector $\hat{j}$ is used). Here, negative sign implies that the force is acting vertically downwards. In order to move the body without acceleration (or with constant velocity), an external applied force $\vec{F}_a$ equal in magnitude but opposite to that of gravitational force $\vec{F}_g$ has to be applied on the body i.e., $\vec{F}_a = -\vec{F}_g$. This implies that $\vec{F}_a = +mg\,\hat{j}$. The positive sign implies that the applied force is in vertically upward direction. Hence, when the body is lifted up its velocity remains unchanged and thus its kinetic energy also remains constant.

The gravitational potential energy (U) at some height *h* is equal to the amount of work required to take the object from the ground to that height *h*.

$$U = \int \vec{F}_a . d\vec{r} = \int_0^h |\vec{F}_a|\ |d\vec{r}| \cos\theta \quad (4.17)$$

Since the displacement and the applied force are in the same upward direction, the angle between them, $\theta = 0°$. Hence, $\cos 0^0 = 1$ and $|\vec{F}_a| = mg$ and $|d\vec{r}| = dr$.

$$U = mg\int_0^h dr \quad (4.18)$$

$$U = mg[r]_0^h = mgh \quad (4.19)$$

Note that the potential energy stored in the object is defined through work done by the external force which is positive. Physically this implies that the agency which is applying the external force is transferring the energy to the object which is then stored as potential energy. If the object is allowed to fall from a height *h* then the stored potential energy is converted into kinetic energy.

Note

- **How can an object move with zero acceleration (constant velocity) when the external force is acting on the object?**

  It is possible when there is another force which acts exactly opposite to the external applied force. They both cancel each other and the resulting net force becomes zero, hence the object moves with zero acceleration.

- Why should the object be moved at constant velocity when we define potential energy?

  If the object does not move at constant velocity, then it will have different velocities at the initial and final locations. According to work-kinetic energy theorem, the external force will impart some extra kinetic energy. But we associate potential energy to the forces like gravitational force, spring force and coulomb force. So the external agency should not impart any kinetic energy when the object is taken from initial to final location.

EXAMPLE 4.8

An object of mass 2 kg is taken to a height 5 m from the ground $\left(g = 10\,\text{m s}^{-2}\right)$.

(a) Calculate the potential energy stored in the object.

(b) Where does this potential energy come from?

(c) What external force must act to bring the mass to that height?

(d) What is the net force that acts on the object while the object is taken to the height 'h'?

Solution

(a) The potential energy $U = mgh$ = $2 \times 10 \times 5 = 100$ J

Here the positive sign implies that the energy is stored on the mass.

(b) This potential energy is transferred from external agency which applies the force on the mass.

(c) The external applied force $\vec{F}_a$ which takes the object to the height 5 m is $\vec{F}_a = -\vec{F}_g$

$$\vec{F}_a = -\left(-mg\hat{j}\right) = mg\hat{j}$$

where, $\hat{j}$ represents unit vector along vertical upward direction.

(d) From the definition of potential energy, the object must be moved at constant velocity. So the net force acting on the object is zero.

$$\vec{F}_g + \vec{F}_a = 0$$

4.2.6 Elastic Potential Energy

When a spring is elongated, it develops a restoring force. *The potential energy possessed by a spring due to a deforming force which stretches or compresses the spring is termed as elastic potential energy*. The work done by the applied force against the restoring force of the spring is stored as the elastic potential energy in the spring.

Consider a spring-mass system. Let us assume a mass, *m* lying on a smooth horizontal table as shown in Figure 4.9. Here, $x = 0$ is the equilibrium position. One end of the spring is attached to a rigid wall and the other end to the mass.

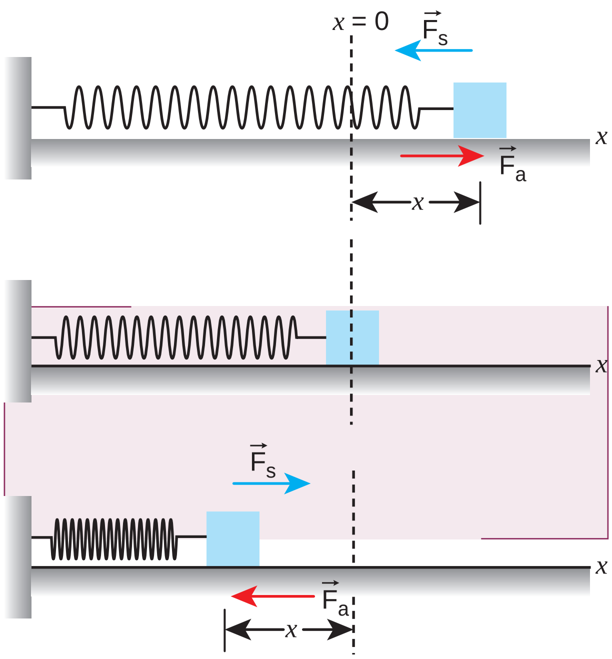


Figure 4.9 Potential energy of the spring (elastic potential energy)

As long as the spring remains in equilibrium position, its potential energy is zero. Now an external force $\vec{F}_a$ is applied so that it is stretched by a distance (x) in the direction of the force.

There is a restoring force called spring force $\vec{F}_s$ developed in the spring which tries to bring the mass back to its original position. This applied force and the spring force are equal in magnitude but opposite in direction i.e., $\vec{F}_a = -\vec{F}_s$. According Hooke's law, the restoring force developed in the spring is

$$\vec{F}_s = -k\vec{x} \qquad (4.20)$$

The negative sign in the above expression implies that the spring force is always opposite to that of displacement $\vec{x}$ and k

is the force constant. Therefore applied force is $\vec{F}_a = +k\vec{x}$. The positive sign implies that the applied force is in the direction of displacement $\vec{x}$. The spring force is an example of variable force as it depends on the displacement $\vec{x}$. Let the spring be stretched to a small distance $d\vec{x}$. The work done by the applied force on the spring to stretch it by a displacement $\vec{x}$ is stored as elastic potential energy.

$$U = \int \vec{F}_a . d\vec{r} = \int_0^x |\vec{F}_a| \, |d\vec{r}| \cos\theta$$
$$= \int_0^x F_a \mathrm{d}x \cos\theta \qquad (4.21)$$

The applied force $\vec{F}_a$ and the displacement $d\vec{r}$ (i.e., here dx) are in the same direction. As, the initial position is taken as the equilibrium position or mean position, x=0 is the lower limit of integration.

$$U = \int_0^x kx\mathrm{d}x \qquad (4.22)$$

$$U = k\left[\frac{x^2}{2}\right]_0^x \qquad (4.23)$$

$$U = \frac{1}{2}kx^2 \qquad (4.24)$$

If the initial position is not zero, and if the mass is changed from position x_i to x_f, then the elastic potential energy is

$$U = \frac{1}{2}k\left(x_f^2 - x_i^2\right) \qquad (4.25)$$

From equations (4.24) and (4.25), we observe that the potential energy of the stretched spring depends on the force constant k and elongation or compression x.

Note The potential energy stored in the spring does not depend on the mass that is attached to the spring.

Force-displacement graph for a spring

Since the restoring spring force and displacement are linearly related as $F = -k\,x$, and are opposite in direction, the graph between F and x is a straight line with dwelling only in the second and fourth quadrant as shown in Figure 4.10. The elastic potential energy can be easily calculated by drawing a F - x graph. The shaded area (triangle) is the work done by the spring force.

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\times(x)\times(kx).$$
$$= \frac{1}{2}kx^2$$

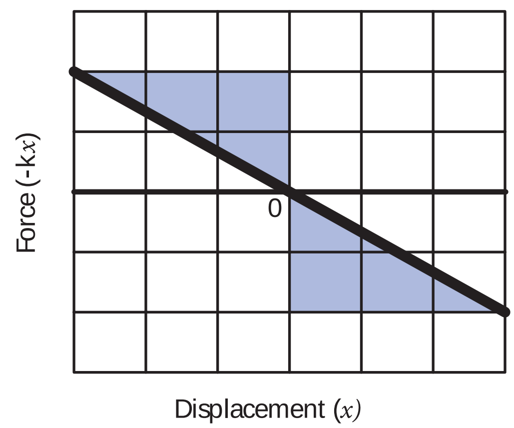


Figure 4.10 Force–displacement graph for a spring

Potential energy-displacement graph for a spring

A compressed or extended spring will transfer its stored potential energy into kinetic energy of the mass attached to the spring. The potential energy-displacement graph is shown in Figure 4.11.

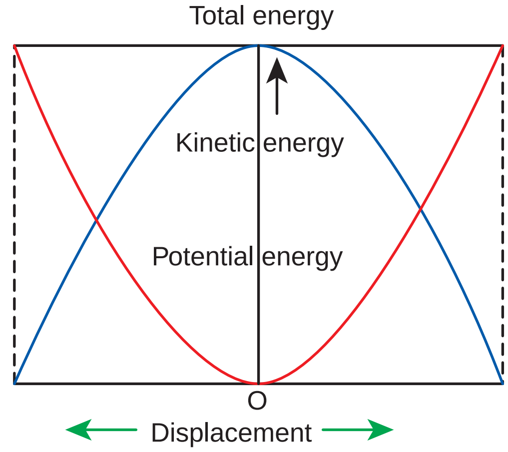


Figure 4.11 Potential energy–displacement graph for a spring-mass system

In a frictionless environment, the energy gets transferred from kinetic to potential and potential to kinetic repeatedly such that the total energy of the system remains constant. At the mean position,

$$\Delta KE = \Delta U \qquad (4.26)$$

EXAMPLE 4.9

Let the two springs A and B be such that $k_A > k_B$. On which spring will more work has to be done if they are stretched by the same force?

Solution

$$F = k_A x_A = k_B x_B$$

$$x_A = \frac{F}{k_A}; \quad x_B = \frac{F}{k_B}$$

The work done on the springs are stored as potential energy in the springs.

$$U_A = \frac{1}{2} k_A x_A^2 \; ; \quad U_B = \frac{1}{2} k_B x_B^2$$

$$\frac{U_A}{U_B} = \frac{k_A x_A^2}{k_B x_B^2} = \frac{k_A \left(\frac{F}{k_A}\right)^2}{k_B \left(\frac{F}{k_B}\right)^2} = \frac{\frac{1}{k_A}}{\frac{1}{k_B}}$$

$$\frac{U_A}{U_B} = \frac{k_B}{k_A}$$

$k_A > k_B$ implies that $U_B > U_A$. Thus, more work is done on B than A.

EXAMPLE 4.10

A body of mass m is attached to the spring which is elongated to 25 cm by an applied force from its equilibrium position.

(a) Calculate the potential energy stored in the spring-mass system?

(b) What is the work done by the spring force in this elongation?

(c) Suppose the spring is compressed to the same 25 cm, calculate the potential energy stored and also the work done by the spring force during compression. (The spring constant, $k = 0.1\ \text{N m}^{-1}$).

Solution

The spring constant, $k = 0.1\ \text{N m}^{-1}$

The displacement, $x = 25\ \text{cm} = 0.25\ \text{m}$

(a) The potential energy stored in the spring is given by

$$U = \frac{1}{2} kx^2 = \frac{1}{2} \times 0.1 \times (0.25)^2 = 0.0031\,\text{J}$$

(b) The work done W_s by the spring force $\vec{F}_s$ is given by,

$$W_s = \int_0^x \vec{F}_s \cdot d\vec{r} = \int_0^x \left(-k x \hat{i}\right) \cdot (dx \hat{i})$$

The spring force $\vec{F}_s$ acts in the negative x direction while elongation acts in the positive x direction.

$$W_s = \int_0^x (-kx)\,dx = -\frac{1}{2}kx^2$$

$$W_s = -\frac{1}{2} \times 0.1 \times (0.25)^2 = -0.0031\,J$$

Note that the potential energy is defined through the work done by the external agency. The positive sign in the potential energy implies that the energy is transferred from the agency to the object. But the work done by the restoring force in this case is negative since restoring force is in the opposite direction to the displacement direction.

(c) During compression also the potential energy stored in the object is the same.

$$U = \frac{1}{2}kx^2 = 0.0031\,J.$$

Work done by the restoring spring force during compression is given by

$$W_s = \int_0^x \vec{F}_s . d\vec{r} = \int_0^x \left(k x \hat{i}\right) . \left(-dx \hat{i}\right)$$

In the case of compression, the restoring spring force acts towards positive x-axis and displacement is along negative x direction.

$$W_s = \int_0^x (-kx)\,dx = -\frac{1}{2}kx^2 = -0.0031\,J$$

4.2.7 Conservative and non-conservative forces

Conservative force

A force is said to be a conservative force if the work done by or against the force in moving the body depends only on the initial and final positions of the body and not on the nature of the path followed between the initial and final positions.

Let us consider an object at point A on the Earth. It can be taken to another point B at a height h above the surface of the Earth by three paths as shown in Figure 4.12.

Whatever may be the path, the work done against the gravitational force is the same as long as the initial and final positions are the same. This is the reason why gravitational force is a conservative force. Conservative force is equal to the negative gradient of the potential energy. In one dimensional case,

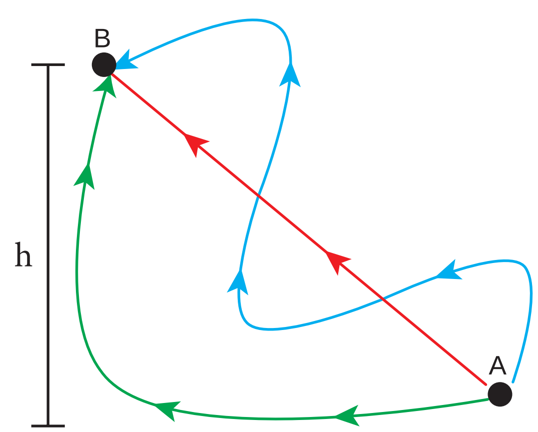


Figure 4.12 Conservative force

$$F_x = -\frac{dU}{dx} \qquad (4.27)$$

Examples for conservative forces are elastic spring force, electrostatic force, magnetic force, gravitational force, etc.

Table 4.3 Comparison of conservative and non-conservative forces

S.No	Conservative forces	Non-conservative forces
1.	Work done is independent of the path	Work done depends upon the path
2.	Work done in a round trip is zero	Work done in a round trip is not zero
3	Total energy remains constant	Energy is dissipated as heat energy
4	Work done is completely recoverable	Work done is not completely recoverable.
5	Force is the negative gradient of potential energy	No such relation exists.

Non-conservative force

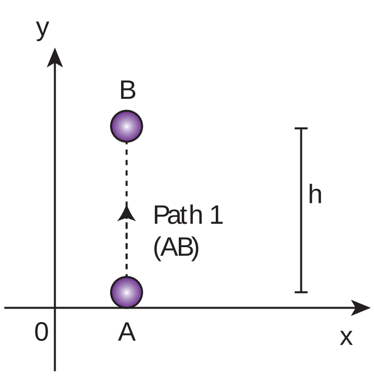
A force is said to be non-conservative if the work done by or against the force in moving a body depends upon the path between the initial and final positions. This means that the value of work done is different in different paths.

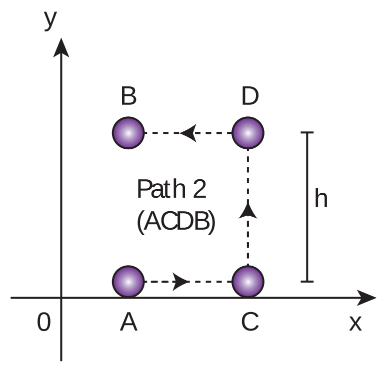
1. Frictional forces are non-conservative forces as the work done against friction depends on the length of the path moved by the body.
2. The force due to air resistance, viscous force are also non-conservative forces as the work done by or against these forces depends upon the velocity of motion.

The properties of conservative and non-conservative forces are summarized in the Table 4.3.

EXAMPLE 4.11

Compute the work done by the gravitational force for the following cases





Solution

Force $\vec{F} = mg\left(-\hat{j}\right) = -mg\hat{j}$

Displacement vector $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

(As the displacement is in two dimension; unit vectors $\hat{i}$ and $\hat{j}$ are used)

(a) Since the motion is only vertical, horizontal displacement component dx is zero. Hence, work done by the force along path 1 (of distance h).

$$W_{\text{path 1}} = \int_A^B \vec{F}.d\vec{r} = \int_A^B \left(-mg\hat{j}\right).\left(dy\,\hat{j}\right)$$

$$= -mg\int_0^h dy = -mgh$$

Total work done for path 2 is

$$W_{\text{path 2}} = \int_A^B \vec{F}.d\vec{r} = \int_A^C \vec{F}.d\vec{r} + \int_C^D \vec{F}.d\vec{r} + \int_D^B \vec{F}.d\vec{r}$$

But

$$\int_A^C \vec{F}.d\vec{r} = \int_A^C \left(-mg\hat{j}\right).\left(dx\hat{i}\right) = 0$$

$$\int_C^D \vec{F}.d\vec{r} = \int_C^D \left(-mg\hat{j}\right).\left(dy\hat{j}\right)$$

$$= -mg\int_0^h dy = -mgh$$

$$\int_D^B \vec{F}.d\vec{r} = \int_D^B \left(-mg\hat{j}\right).\left(-dx\hat{i}\right) = 0$$

Therefore, the total work done by the force along the path 2 is

$$W_{\text{path 2}} = \int_A^B \vec{F}.d\vec{r} = -mgh$$

Note that the work done by the conservative force is independent of the path.

EXAMPLE 4.12

Consider an object of mass 2 kg moved by an external force 20 N in a surface having coefficient of kinetic friction 0.9 to a distance 10 m. What is the work done by the external force and kinetic friction ? Comment on the result. (Assume g = 10 ms^{-2})

Solution

m = 2 kg, d = 10 m, F_{ext} = 20 N, μ_k = 0.9.

When an object is in motion on the horizontal surface, it experiences two forces.

(a) External force, $F_{ext} = 20$ N

(b) Kinetic friction,
$f_k = \mu_k mg = 0.9 \times (2) \times 10 = 18\,\text{N}$.

The work done by the external force $W_{ext} = F\,d = 20 \times 10 = 200\,\text{J}$

The work done by the force of kinetic friction $W_k = f_k d = (-18) \times 10 = -180$ J. Here the negative sign implies that the force of kinetic friction is opposite to the direction of displacement.

The total work done on the object $W_{total} = W_{ext} + W_k = 200\text{ J} - 180\text{ J} = 20\text{ J}$.

Since the friction is a non-conservative force, out of 200 J given by the external force, the 180 J is lost and it can not be recovered.

4.2.8 Law of conservation of energy

When an object is thrown upwards its kinetic energy goes on decreasing and consequently its potential energy keeps increasing (neglecting air resistance). When it reaches the highest point its energy is completely potential. Similarly, when the object falls back from a height its kinetic energy increases whereas its potential energy decreases. When it touches the ground its energy is completely kinetic. At the intermediate points the energy is both kinetic and potential as shown in Figure 4.13. When the body reaches the ground the kinetic energy is completely dissipated into some other form of energy like sound, heat, light and deformation of the body etc.

In this example the energy transformation takes place at every point. The sum of kinetic energy and potential energy i.e., the total mechanical energy always remains constant, implying that the total energy is conserved. This is stated as the law of conservation of energy.

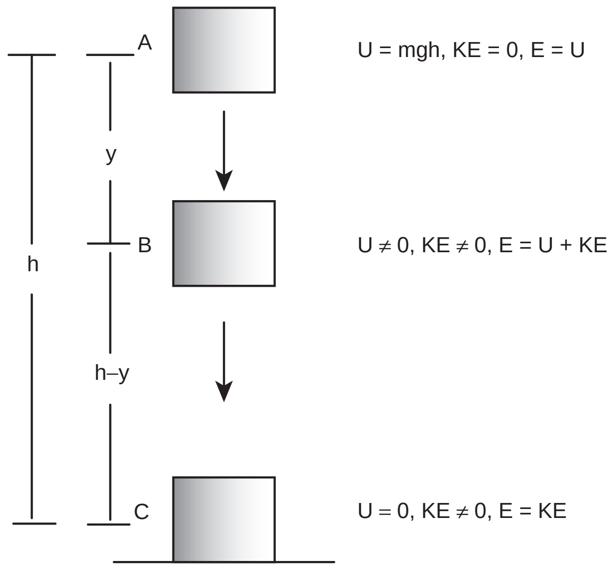


Figure 4.13 Conservation of energy

The law of conservation of energy states that *energy can neither be created nor destroyed. It may be transformed from one form to another but the total energy of an isolated system remains constant.*

Figure 4.13 illustrates that, if an object starts from rest at height h, the total energy is purely potential energy (U=mgh) and the kinetic energy (KE) is zero at h. When the object falls at some distance y, the potential energy and the kinetic energy are not zero whereas, the total energy remains same as measured at height h. When the object is about to touch the ground, the potential energy is zero and total energy is purely kinetic.

EXAMPLE 4.13

An object of mass 1 kg is falling from the height h = 10 m. Calculate

(a) The total energy of an object at $h = 10$ m
(b) Potential energy of the object when it is at $h = 4$ m
(c) Kinetic energy of the object when it is at $h = 4$ m
(d) What will be the speed of the object when it hits the ground?

(Assume $g = 10$ m s^{-2})

Solution

(a) The gravitational force is a conservative force. So the total energy remains constant throughout the motion. At $h = 10$ m, the total energy E is entirely potential energy.

$$E = U = mgh = 1 \times 10 \times 10 = 100 \text{ J}$$

(b) The potential energy of the object at $h = 4$ m is

$$U = mgh = 1 \times 10 \times 4 = 40 \text{ J}$$

(c) Since the total energy is constant throughout the motion, the kinetic energy at $h = 4$ m must be $KE = E - U = 100 - 40 = 60$ J

Alternatively, the kinetic energy could also be found from velocity of the object at 4 m. At the height 4 m, the object has fallen through a height of 6 m.

The velocity after falling 6 m is calculated from the equation of motion,

$$\text{v} = \sqrt{2gh} = \sqrt{2 \times 10 \times 6} = \sqrt{120} \text{ m s}^{-1};$$
$$\text{v}^2 = 120$$

The kinetic energy is $\text{KE} = \frac{1}{2}\text{mv}^2 = \frac{1}{2} \times 1 \times 120 = 60$ J

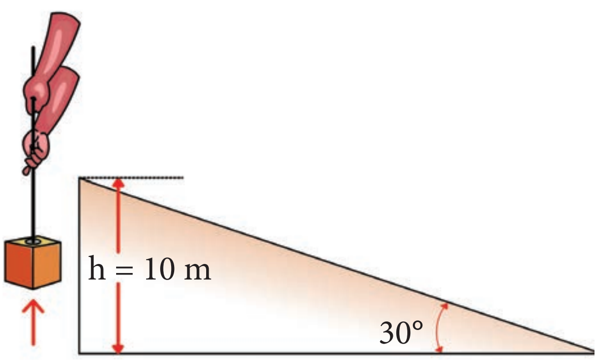
(d) When the object is just about to hit the ground, the total energy is completely kinetic and the potential energy, $U = 0$.

$$E = KE = \frac{1}{2}mv^2 = 100\ J$$

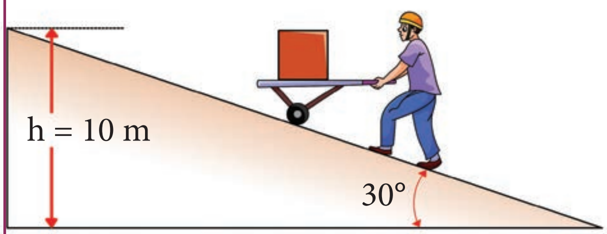
$$v = \sqrt{\frac{2}{m}KE} = \sqrt{\frac{2}{1}\times 100} = \sqrt{200} \approx 14.12\ m\ s^{-1}$$

EXAMPLE 4.14

A body of mass 100 kg is lifted to a height 10 m from the ground in two different ways as shown in the figure. What is the work done by the gravity in both the cases? Why is it easier to take the object through a ramp?



Path (1) straight up



Path (2) along the ramp

Solution

m = 100 kg, h = 10 m

Along path (1):

The minimum force F_1 required to move the object to the height of 10 m should be equal to the gravitational force, $F_1 = mg = 100 \times 10 = 1000$ N

The distance moved along path (1) is, $h = 10$ m

The work done on the object along path (1) is

$$W = F\ h = 1000 \times 10 = 10{,}000\ J$$

Along path (2):

In the case of the ramp, the minimum force F_2 that we apply on the object to take it up is not equal to mg, it is rather equal to $mg\sin\theta$. $(mg\sin\theta < mg)$.

Here, angle θ = 30°

Therefore, $F_2 = mg\sin\theta = 100 \times 10 \times \sin 30° = 100 \times 10 \times 0.5 = 500$ N

Hence, $(mg\sin\theta < mg)$

The path covered along the ramp is,

$$l = \frac{h}{\sin 30°} = \frac{10}{0.5} = 20\,m$$

The work done on the object along path (2) is, $W = F_2\, l = 500 \times 20 = 10{,}000$ J

Since the gravitational force is a conservative force, the work done by gravity on the object is independent of the path taken.

In both the paths the work done by the gravitational force is 10,000 J

Along path (1): more force needs to be applied against gravity to cover lesser distance .

Along path (2): lesser force needs to be applied against the gravity to cover more distance.

As the force needs to be applied along the ramp is less, it is easier to move the object along the ramp.

EXAMPLE 4.15

An object of mass m is projected from the ground with initial speed v_0.

Find the speed at height h.

Solution

Since the gravitational force is conservative; the total energy is conserved throughout the motion.

	Initial	Final
Kinetic energy	$\frac{1}{2}mv_0^2$	$\frac{1}{2}mv^2$
Potential energy	0	mgh
Total energy	$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_0^2$	$\frac{1}{2}mv^2 + mgh$

Final values of potential energy, kinetic energy and total energy are measured at the height *h*.

By law of conservation of energy, the initial and final total energies are the same.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgh$$

$$v_0^2 = v^2 + 2gh$$

$$v = \sqrt{v_0^2 - 2gh}$$

Note that in section (2.11.2) similar result is obtained using kinematic equation based on calculus method. However, calculation through energy conservation method is much easier than calculus method.

EXAMPLE 4.16

An object of mass 2 kg attached to a spring is moved to a distance $x = 10$ m from its equilibrium position. The spring constant $k = 1$ N m^{-1} and assume that the surface is frictionless.

(a) When the mass crosses the equilibrium position, what is the speed of the mass?

(b) What is the force that acts on the object when the mass crosses the equilibrium position and extremum position $x = \pm 10$ m.

Solution

(a) Since the spring force is a conservative force, the total energy is constant. At $x = 10$ m, the total energy is purely potential.

$$E = U = \frac{1}{2}kx^2 = \frac{1}{2}\times(1)\times(10)^2 = 50 \text{ J}$$

When the mass crosses the equilibrium position $(x = 0)$, the potential energy

$$U = \frac{1}{2}\times 1\times(0) = 0 \text{ J}$$

The entire energy is purely kinetic energy at this position.

$$E = KE = \frac{1}{2}mv^2 = 50 \text{ J}$$

The speed

$$v = \sqrt{\frac{2KE}{m}} = \sqrt{\frac{2\times 50}{2}} = \sqrt{50}\, m s^{-1} \approx 7.07\, m s^{-1}$$

(b) Since the restoring spring force is F = - kx, when the object crosses the equilibrium position, it experiences no force. Note that at equilibrium position, the object moves very fast. When the object is at x = +10 m (elongation), the force F = – k x

F = – (1) (10) = – 10 N. Here the negative sign implies that the force is towards equilibrium i.e., towards negative x-axis and when the object is at $x = -10m$ (compression), it experiences a forces F = – (1) (– 10) = +10 N. Here the positive sign implies that the force points towards positive x-axis.

The object comes to momentary rest at $x = \pm 10\,m$ even though it experiences a maximum force at both these points.

4.2.9 Motion in a vertical circle

Imagine that a body of mass (m) attached to one end of a massless and inextensible string executes circular motion in a vertical plane with the other end of the string fixed. The length of the string becomes the radius ($\vec{r}$) of the circular path (Figure 4.14).

Let us discuss the motion of the body by taking the free body diagram (FBD) at a position where the position vector ($\vec{r}$) makes an angle θ with the vertically downward direction and the instantaneous velocity is as shown in Figure 4.14.

There are two forces acting on the mass.

1. Gravitational force which acts downward
2. Tension along the string.

Applying Newton's second law on the mass,

In the tangential direction,

$$mg\sin\theta = m a_t$$
$$mg\sin\theta = -m\left(\frac{dv}{dt}\right) \quad (4.28)$$

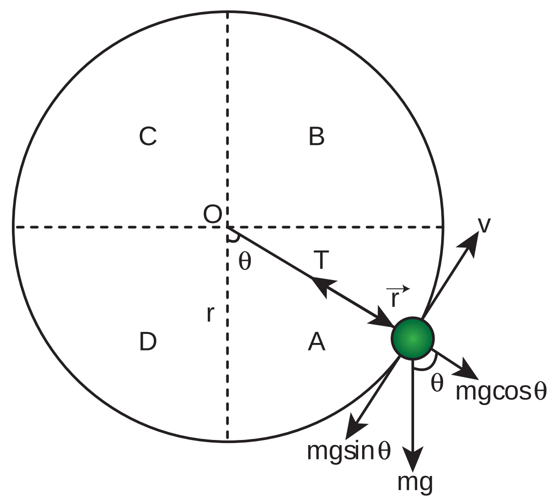


Figure 4.14 Motion in vertical circle

where, $a_t = -\frac{dv}{dt}$ is tangential retardation

In the radial direction,

$$T - mg\cos\theta = m a_r$$
$$T - mg\cos\theta = \frac{mv^2}{r} \quad (4.29)$$

where, $a_r = \frac{v^2}{r}$ is the centripetal acceleration.

The circle can be divided into four sections A, B, C, D for better understanding of the motion. The four important facts to be understood from the two equations are as follows:

(i) The mass is having tangential acceleration (g sin θ) for all values of θ (except θ = 0°), it is clear that this vertical cirular motion is not a uniform circular motion.

(ii) From the equations (4.28) and (4.29) it is understood that as the magnitude of velocity is not a constant in the course of motion, the tension in the string is also not constant.

(iii) The equation (4.29), $T = mg\cos\theta + \frac{mv^2}{r}$ highlights that in sections A and D of the circle, $\left(\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2};\ \cos\theta \text{ is positive}\right)$, the term mg cosθ is always greater than zero. Hence the tension cannot vanish even when the velocity vanishes.

(iv) The equation (4.29), $\frac{mv^2}{r} = T - mg\cos\theta$; further highlights that in sections B and C of the circle, $\left(\text{for } \frac{\pi}{2} < \theta < \frac{3\pi}{2};\ \cos\theta \text{ is negative}\right)$, the second term is always greater than zero. Hence velocity cannot vanish, even when the tension vanishes.

These points are to be kept in mind while solving problems related to motion in vertical circle.

To start with let us consider only two positions, say the lowest point 1 and the highest point 2 as shown in Figure 4.15 for further analysis. Let the velocity of the body at the lowest point 1 be $\vec{v}_1$, at the highest point 2 be $\vec{v}_2$ and $\vec{v}$ at any other point. The direction of velocity is tangential to the circular path at all points. Let $\vec{T}_1$ be the tension in the string at the lowest point and $\vec{T}_2$ be the tension at the highest point and $\vec{T}$ be the tension at any other point. Tension at each point acts towards the centre. The tensions and velocities at these two points can be found by applying the law of conservation of energy.

For the lowest point (1)

When the body is at the lowest point 1, the gravitational force $m\vec{g}$ which acts on the body (vertically downwards) and another one is the tension $\vec{T}_1$ acting vertically upwards, i.e. towards the centre. From the equation (4.29), we get

$$T_1 - mg = \frac{mv_1^2}{r} \qquad (4.30)$$

$$T_1 = \frac{mv_1^2}{r} + mg \qquad (4.31)$$

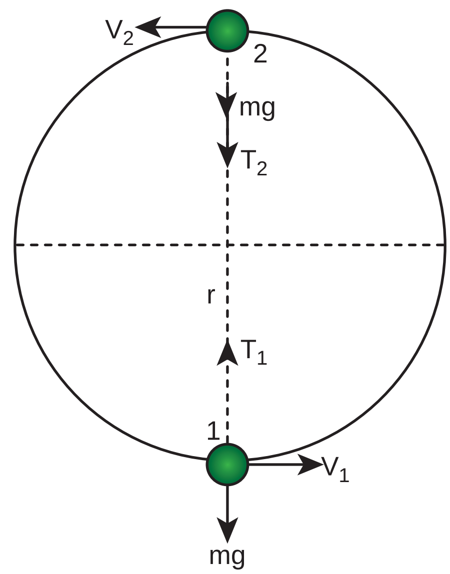


Figure 4.15 Motion in vertical circle shown for lowest and highest points

For the highest point (2)

At the highest point 2, both the gravitational force $m\vec{g}$ on the body and the tension $\vec{T}_2$ act downwards, i.e. towards the centre again.

$$T_2 + mg = \frac{mv_2^2}{r} \qquad (4.32)$$

$$T_2 = \frac{mv_2^2}{r} - mg \qquad (4.33)$$

From equations (4.31) and (4.33), it is understood that $T_1 > T_2$. The difference in tension $T_1 - T_2$ is obtained by subtracting equation (4.33) from equation (4.31).

$$T_1 - T_2 = \frac{mv_1^2}{r} + mg - \left(\frac{mv_2^2}{r} - mg\right)$$

$$= \frac{mv_1^2}{r} + mg - \frac{mv_2^2}{r} + mg$$

$$T_1 - T_2 = \frac{m}{r}\left[v_1^2 - v_2^2\right] + 2mg \qquad (4.34)$$

The term $\left[v_1^2 - v_2^2\right]$ can be found easily by applying law of conservation of energy at point 1 and also at point 2.

Note

The tension will not do any work on the mass as the tension and the direction of motion is always perpendicular.

The gravitational force is doing work on the mass, as it is a conservative force the total energy of the mass is conserved throughout the motion.

Total Energy at point 1(E_1) is same as the total energy at a point 2 (E_2)

$$E_1 = E_2 \qquad (4.35)$$

Potential Energy at point 1, $U_1 = 0$ (by taking reference as point 1)

Kinetic Energy at point 1, $KE_1 = \frac{1}{2}mv_1^2$

Total Energy at point 1, $E_1 = U_1 + KE_1 =$

$= 0 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_1^2$

Similarly, Potential Energy at point 2, $U_2 = mg\ (2r)$ (h is 2r from point 1)

Kinetic Energy at point 2, $KE_2 = \frac{1}{2}mv_2^2$

Total Energy at point 2, $E_2 = U_2 + KE_2 =$

$2\ mg\ r + \frac{1}{2}mv_2^2$

From the law of conservation of energy given in equation (4.35), we get

$$\frac{1}{2}mv_1^2 = 2mgr + \frac{1}{2}mv_2^2$$

After rearranging,

$$\frac{1}{2}m\left(v_1^2 - v_2^2\right) = 2mgr$$

$$v_1^2 - v_2^2 = 4gr \qquad (4.36)$$

Substituting equation (4.36) in equation (4.34) we get,

$$T_1 - T_2 = \frac{m}{r}\left[4gr\right] + 2mg$$

Therefore, the difference in tension is

$$T_1 - T_2 = 6\ mg \qquad (4.37)$$

Minimum speed at the highest point (2)

The body must have a minimum speed at point 2 otherwise, the string will slack before reaching point 2 and the body will not loop the circle. To find this minimum speed let us take the tension $T_2 = 0$ in equation (4.33).

$$0 = \frac{mv_2^2}{r} - mg$$

$$\frac{mv_2^2}{r} = mg$$

$$v_2^2 = rg$$

$$v_2 = \sqrt{gr} \qquad (4.38)$$

The body must have a speed at point 2, $v_2 \geq \sqrt{gr}$ to stay in the circular path.

Minimum speed at the lowest point 1

To have this minimum speed $\left(v_2 = \sqrt{gr}\right)$ at point 2, the body must have minimum speed also at point 1.

By making use of equation (4.36) we can find the minimum speed at point 1.

$$v_1^2 - v_2^2 = 4gr$$

Substituting equation (4.38) in (4.36),

$$v_1^2 - gr = 4gr$$

$$v_1^2 = 5gr$$

$$v_1 = \sqrt{5gr} \qquad (4.39)$$

The body must have a speed at point 1, $v_1 \geq \sqrt{5gr}$ to stay in the circular path.

From equations (4.38) and (4.39), it is clear that the minimum speed at the lowest point 1 should be $\sqrt{5}$ times more than the minimum speed at the highest point 2, so that the body loops without leaving the circle.

EXAMPLE 4.17

Water in a bucket tied with rope is whirled around in a vertical circle of radius 0.5 m. Calculate the minimum velocity at the lowest point so that the water does not spill from it in the course of motion. ($g = 10\ ms^{-2}$)

Solution

Radius of circle r = 0.5 m

The required speed at the highest point $v_2 = \sqrt{gr} = \sqrt{10 \times 0.5} = \sqrt{5}\ ms^{-1}$. The speed at lowest point $v_1 = \sqrt{5gr} = \sqrt{5} \times \sqrt{gr} = \sqrt{5} \times \sqrt{5} = 5\ ms^{-1}$

7MX58

4.3 POWER

4.3.1 Definition of power

Power is a measure of how fast or slow a work is done. *Power is defined as the rate of work done or energy delivered.*

$$\text{Power } (P) = \frac{\text{work done}(W)}{\text{time taken}(t)}$$

$$P = \frac{W}{t}$$

Average power

The average power (P_{av}) is defined as the ratio of the total work done to the total time taken.

$$P_{av} = \frac{\text{total work done}}{\text{total time taken}}$$

Instantaneous power

The instantaneous power (P_{inst}) is defined as the power delivered at an instant (as time interval approaches zero),

$$P_{inst} = \frac{dW}{dt}$$

4.3.2 Unit of power

Power is a scalar quantity. Its dimension is $[ML^2T^{-3}]$. The SI unit of power is watt (W), named after the inventor of the steam engine James Watt. *One watt is defined as* the power *when one joule of work is done in one second*, ($1\ W = 1\ J\ s^{-1}$).

The higher units are kilowatt(kW), megawatt(MW), and Gigawatt(GW).

$$1kW = 1000\ W = 10^3\ watt$$
$$1MW = 10^6\ watt$$
$$1GW = 10^9\ watt$$

For motors, engines and some automobiles an old unit of power still commercially in use which is called as the horse-power (hp). We have a conversion for horse-power (hp) into watt (W) which is,

$$1\ hp = 746\ W$$

All electrical goods come with a definite power rating in watt printed on them. A 100 watt bulb consumes 100 joule of electrical energy in one second. The energy measured in joule in terms of power in watt and time in second is written as, 1 J =1 W s. When electrical appliances are put in use for long hours, they consume a large amount of energy. Measuring the electrical energy in a small unit watt. second (W s) leads to handling large numerical values. Hence, electrical energy is measured in the unit called kilowatt hour (kWh).

$$1\ \text{electrical unit} = 1\ kWh = 1 \times (10^3\ W) \times (3600\ s)$$
$$1\ \text{electrical unit} = 3600 \times 10^3\ W\ s$$
$$1\ \text{electrical unit} = 3.6 \times 10^6\ J$$
$$1\ kWh = 3.6 \times 10^6\ J$$

Electricity bills are generated in units of kWh for electrical energy consumption. 1 unit of electrical energy is 1 kWh. (**Note**: kWh is unit of energy and not of power.)

EXAMPLE 4.18

Calculate the energy consumed in electrical units when a 75 W fan is used for 8 hours daily for one month (30 days).

Solution

Power, P = 75 W

Time of usage, t = 8 hour × 30 days = 240 hours

Electrical energy consumed is the product of power and time of usage.

Electrical energy = power × time of usage = P × t

$$= 75 \text{ watt} \times 240 \text{ hour}$$
$$= 18000 \text{ watt hour}$$
$$= 18 \text{ kilowatt hour} = 18\,\text{kWh}$$
$$1 \text{ electrical unit} = 1 \text{ kWh}$$
$$\text{Electrical energy} = 18 \text{ unit}$$

DO YOU KNOW?

Incandescent lamps glow for 1000 hours. CFL lamps glow for 6000 hours. But LED lamps glow for 50000 hrs (almost 25 years at 5.5 hour per day).

4.3.3 Relation between power and velocity

The work done by a force $\vec{F}$ for a displacement $d\vec{r}$ is

$$W = \int \vec{F} \cdot d\vec{r} \qquad (4.40)$$

Left hand side of the equation (4.40) can be written as

$$W = \int dW = \int \frac{dW}{dt} dt$$

(multiplied and divided by dt)

(4.41)

Since, velocity is $\vec{v} = \frac{d\vec{r}}{dt}$; $d\vec{r} = \vec{v}\,dt$. Right hand side of the equation (4.40) can be written as

$$\int \vec{F} \cdot d\vec{r} = \int \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \int \left(\vec{F} \cdot \vec{v} \right) dt \quad \left[\vec{v} = \frac{d\vec{r}}{dt} \right] \qquad (4.42)$$

Substituting equation (4.41) and equation (4.42) in equation (4.40), we get

$$\int \frac{dW}{dt} dt = \int \left(\vec{F} \cdot \vec{v} \right) dt$$
$$\int \left(\frac{dW}{dt} - \vec{F} \cdot \vec{v} \right) dt = 0$$

This relation is true for any arbitrary value of dt. This implies that the term within the bracket must be equal to zero, i.e.,

$$\frac{dW}{dt} - \vec{F} \cdot \vec{v} = 0$$

Or

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = P \qquad (4.43)$$

EXAMPLE 4.19

A vehicle of mass 1250 kg is driven with an acceleration 0.2 m s^{-2} along a straight level road against an external resistive force 500 N. Calculate the power delivered by the vehicle's engine if the velocity of the vehicle is 30 m s^{-1}.

Solution

The vehicle's engine has to do work against resistive force and make vechile to move with an acceleration. Therefore, power delivered by the vehicle engine is

P = (resistive force + mass × acceleration) (velocity)

$$P = \vec{F}_{-tot} \cdot \vec{v} = (F_{resistive} + F)\vec{v}$$
$$P = \vec{F}_{tot} \cdot \vec{v} = (F_{resistive} + ma)\vec{v}$$
$$= (500\text{ N} + (1250\text{ kg}) \times (0.2\ ms^{-2}))(30\text{ ms}^{-1}) = 22.5\text{ kW}$$

4.4 COLLISIONS

Collision is a common phenomenon that happens around us every now and then. For example, carom, billiards, marbles, etc.,. Collisions can happen between two bodies with or without physical contacts.

Linear momentum is conserved in all collision processes. When two bodies collide, the mutual impulsive forces acting between them during the collision time (Δt) produces a change in their respective momenta. That is, the first body exerts a force $\vec{F}_{21}$ on the second body. From Newton's third law, the second body exerts a force $\vec{F}_{12}$ on the first body. This causes a change in momentum $\Delta\vec{p}_1$ and $\Delta\vec{p}_2$ of the first body and second body respectively. Now, the relations could be written as,

$$\Delta\vec{p}_1 = \vec{F}_{12}\Delta t \qquad (4.44)$$

$$\Delta\vec{p}_2 = \vec{F}_{21}\Delta t \qquad (4.45)$$

Adding equation (4.44) and equation (4.45), we get

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = \vec{F}_{12}\Delta t + \vec{F}_{21}\Delta t = \left(\vec{F}_{12} + \vec{F}_{21}\right)\Delta t$$

According to Newton's third law, $\vec{F}_{12} = -\vec{F}_{21}$

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$$

$$\Delta\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

Dividing both sides by Δt and taking limit $\Delta t \rightarrow 0$, we get

$$\lim_{\Delta t \to 0} \frac{\Delta\left(\vec{p}_1 + \vec{p}_2\right)}{\Delta t} = \frac{d\left(\vec{p}_1 + \vec{p}_2\right)}{dt} = 0$$

The above expression implies that the total linear momentum is a conserved quantity. **Note:** The momentum is a vector quantity. Hence, vector addition has to be followed to find the total momentum of the individual bodies in collision.

4.4.1 Types of Collisions

In any collision process, the total linear momentum and total energy are always conserved whereas the total kinetic energy need not be conserved always. Some part of the initial kinetic energy is transformed to other forms of energy. This is because, the impact of collisions and deformation occurring due to collisions may in general, produce heat, sound, light etc. By taking these effects into account, we classify the types of collisions as follows:

(a) Elastic collision

(b) Inelastic collision

(a) Elastic collision

In a collision, the total initial kinetic energy of the bodies (before collision) is equal to the total final kinetic energy of the bodies (after collision) then, it is called as elastic collision. i.e.,

Total kinetic energy before collision = Total kinetic energy after collision

(b) Inelastic collision

In a collision, the total initial kinetic energy of the bodies (before collision) is not equal to the total final kinetic energy of the bodies (after collision) then, it is called as inelastic collision. i.e.,

Total kinetic energy before collision ≠ Total kinetic energy after collision

$$\begin{pmatrix}\textit{Total kinetic energy}\\ \textit{before collision}\end{pmatrix} - \begin{pmatrix}\textit{Total kinetic energy}\\ \textit{after collision}\end{pmatrix} = \begin{pmatrix}\textit{loss in energy}\\ \textit{during collision}\end{pmatrix} = \Delta Q$$

Even though kinetic energy is not conserved but the total energy is conserved. This is because the total energy contains the kinetic energy term and also a term ΔQ, which includes all the losses that take place during collision. Note that loss in kinetic energy during collision is transformed to another form of energy like sound, thermal, etc. Further, if the two colliding bodies stick together after collision such collisions are known as completely inelastic collision or perfectly inelastic collision. Such a collision is found very often. For example when a clay putty is thrown on a moving vehicle, the clay putty (or Bubblegum) sticks to the moving vehicle and they move together with the same velocity.

4.4.2 Elastic collisions in one dimension

Consider two elastic bodies of masses m_1 and m_2 moving in a straight line (along positive x direction) on a frictionless horizontal surface as shown in Figure 4.16.

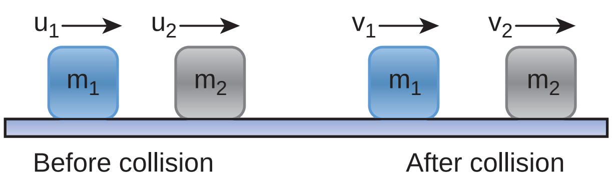


Figure 4.16 Elastic collision in one dimension

Mass	Initial velocity	Final velocity
Mass m_1	u_1	v_1
Mass m_2	u_2	v_2

In order to have collision, we assume that the mass m_1 moves faster than mass m_2 i.e., $u_1 > u_2$. For elastic collision, the total linear momentum and kinetic energies of the two bodies before and after collision must remain the same.

Table 4.4 Comparison between elastic and inelastic collisions

S.No.	Elastic Collision	Inelastic Collision
1.	Total momentum is conserved	Total momentum is conserved
2.	Total kinetic energy is conserved	Total kinetic energy is not conserved
3.	Forces involved are conservative forces	Forces involved are non-conservative forces
4.	Mechanical energy is not dissipated.	Mechanical energy is dissipated into heat, light, sound etc.

	Momentum of mass m_1	Momentum of mass m_2	Total linear momentum
Before collision	$p_{i1} = m_1 u_1$	$p_{i2} = m_2 u_2$	$p_i = p_{i1} + p_{i2}$ $p_i = m_1 u_1 + m_2 u_2$
After collision	$p_{f1} = m_1 v_1$	$p_{f2} = m_2 v_2$	$p_f = p_{f1} + p_{f2}$ $p_f = m_1 v_1 + m_2 v_2$

From the law of conservation of linear momentum,

Total momentum before collision (p_i) = Total momentum after collision (p_f)

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad (4.46)$$

$$\text{Or} \quad m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \quad (4.47)$$

Further,

	Kinetic energy of mass m_1	Kinetic energy of mass m_2	Total kinetic energy
Before collision	$KE_{i1} = \frac{1}{2} m_1 u_1^2$	$KE_{i2} = \frac{1}{2} m_2 u_2^2$	$KE_i = KE_{i1} + KE_{i2}$ $KE_i = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$
After collision	$KE_{f1} = \frac{1}{2} m_1 v_1^2$	$KE_{f2} = \frac{1}{2} m_2 v_2^2$	$KE_f = KE_{f1} + KE_{f2}$ $KE_f = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

For elastic collision,

Total kinetic energy before collision KE_i = Total kinetic energy after collision KE_f

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad (4.48)$$

After simplifying and rearranging the terms,

$$m_1 (u_1^2 - v_1^2) = m_2 (v_2^2 - u_2^2)$$

Using the formula $a^2 - b^2 = (a+b)(a-b)$, we can rewrite the above equation as

$$m_1 (u_1 + v_1)(u_1 - v_1) = m_2 (v_2 + u_2)(v_2 - u_2) \quad (4.49)$$

Dividing equation (4.49) by (4.47) gives,

$$\frac{m_1(u_1+v_1)(u_1-v_1)}{m_1(u_1-v_1)} = \frac{m_2(v_2+u_2)(v_2-u_2)}{m_2(v_2-u_2)}$$

$$u_1 + v_1 = v_2 + u_2$$

Rearranging, (4.50)

$$u_1 - u_2 = v_2 - v_1$$

Equation (4.50) can be rewritten as

$$u_1 - u_2 = -(v_1 - v_2)$$

This means that for any elastic head on collision, the relative speed of the two elastic bodies after the collision has the same magnitude as before collision but in opposite direction. Further note that this result is independent of mass.

Rewriting the above equation for v_1 and v_2,

$$v_1 = v_2 + u_2 - u_1 \quad (4.51)$$

Or

$$v_2 = u_1 + v_1 - u_2 \quad (4.52)$$

To find the final velocities v_1 and v_2:
Substituting equation (4.52) in equation (4.47) gives the velocity of m_1 as

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - u_2 - u_2)$$

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - 2u_2)$$

$$m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1 - 2m_2u_2$$

$$m_1u_1 - m_2u_1 + 2m_2u_2 = m_1v_1 + m_2v_1$$

$$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$

$$\text{or } v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2 \quad (4.53)$$

Similarly, by substituting (4.51) in equation (4.47) or substituting equation (4.53) in equation (4.52), we get the final velocity of m_2 as

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2 \quad (4.54)$$

Case 1: When bodies has the same mass i.e., $m_1 = m_2$,

$$\text{equation (4.53)} \Rightarrow v_1 = (0)\ u_1 + \left(\frac{2m_2}{2m_2}\right)u_2$$

$$v_1 = u_2 \quad (4.55)$$

$$\text{equation (4.54)} \Rightarrow v_2 = \left(\frac{2m_1}{2m_1}\right)u_1 + (0)u_2$$

$$v_2 = u_1 \quad (4.56)$$

The equations (4.55) and (4.56) show that in one dimensional elastic collision, when two bodies of equal mass collide after the collision their velocities are exchanged.

Case 2: When bodies have the same mass i.e., $m_1 = m_2$ and second body (usually called target) is at rest ($u_2 = 0$),

By substituting $m_1 = m_2$ and $u_2 = 0$ in equations (4.53) and equations (4.54) we get,

$$\text{from equation (4.53)} \Rightarrow v_1 = 0 \quad (4.57)$$

$$\text{from equation (4.54)} \Rightarrow v_2 = u_1 \quad (4.58)$$

Equations (4.57) and (4.58) show that when the first body comes to rest the second body moves with the initial velocity of the first body.

Case 3:

The first body is very much lighter than the second body

$\left(m_1 \ll m_2, \frac{m_1}{m_2} \ll 1\right)$ then the ratio $\frac{m_1}{m_2} \approx 0$ and also if the target is at rest $(u_2 = 0)$

Dividing numerator and denominator of equation (4.53) by m_2, we get

$$v_1 = \left(\frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1}\right) u_1 + \left(\frac{2}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_1 = \left(\frac{0-1}{0+1}\right) u_1$$

$$v_1 = -u_1 \qquad (4.59)$$

Similarly,

Dividing numerator and denominator of equation (4.54) by m_2, we get

$$v_2 = \left(\frac{2\frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right) u_1 + \left(\frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_2 = (0)u_1 + \left(\frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_2 = 0 \qquad (4.60)$$

The equation (4.59) implies that the first body which is lighter returns back (rebounds) in the opposite direction with the same initial velocity as it has a negative sign. The equation (4.60) implies that the second body which is heavier in mass continues to remain at rest even after collision. For example, if a ball is thrown at a fixed wall, the ball will bounce back from the wall with the same velocity with which it was thrown but in opposite direction.

Case 4:

The second body is very much lighter than the first body

$\left(m_2 \ll m_1, \frac{m_2}{m_1} \ll 1\right)$ then the ratio $\frac{m_2}{m_1} \approx 0$ and also if the target is at rest $(u_2 = 0)$

Dividing numerator and denominator of equation (4.53) by m_1, we get

$$v_1 = \left(\frac{1 - \frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}\right) u_1 + \left(\frac{2\frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}}\right)(0)$$

$$v_1 = \left(\frac{1-0}{1+0}\right) u_1 + \left(\frac{0}{1+0}\right)(0)$$

$$v_1 = u_1 \qquad (4.61)$$

Similarly,

Dividing numerator and denominator of equation (4.58) by m_1, we get

$$v_2 = \left(\frac{2}{1 + \frac{m_2}{m_1}}\right) u_1 + \left(\frac{\frac{m_2}{m_1} - 1}{1 + \frac{m_2}{m_1}}\right)(0)$$

$$v_2 = \left(\frac{2}{1+0}\right) u_1$$

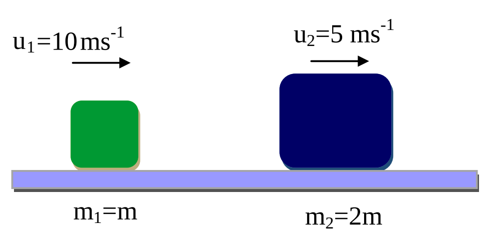
$$v_2 = 2u_1 \qquad (4.62)$$

The equation (4.61) implies that the first body which is heavier continues to move with the same initial velocity. The equation (4.62) suggests that the second body which is lighter will move with twice the initial velocity of the first body. It means that the lighter body is thrown away from the point of collision.

EXAMPLE 4.20

A lighter particle moving with a speed of 10 m s^{-1} collides with an object of double its mass moving in the same direction with half its speed. Assume that the collision is a one dimensional elastic collision. What will be the speed of both particles after the collision?

Solution



Let the mass of the first body be *m* which moves with an initial velocity, $u_1 = 10$ m s^{-1}. Therefore, the mass of second body is *2m* and its initial velocity is $u_2 = \frac{1}{2}u_1 = \frac{1}{2}\left(10\text{ m s}^{-1}\right)$,

Then, the final velocities of the bodies can be calculated from the equation (4.53) and equation (4.54)

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = \left(\frac{m - 2m}{m + 2m}\right)10 + \left(\frac{2 \times 2m}{m + 2m}\right)5$$

$$v_1 = -\left(\frac{1}{3}\right)10 + \left(\frac{4}{3}\right)5 = \frac{-10 + 20}{3} = \frac{10}{3}$$

$$v_1 = 3.33\text{ ms}^{-1}$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = \left(\frac{2m}{m + 2m}\right)10 + \left(\frac{2m - m}{m + 2m}\right)5$$

$$v_2 = \left(\frac{2}{3}\right)10 + \left(\frac{1}{3}\right)5 = \frac{20 + 5}{3} = \frac{25}{3}$$

$$v_2 = 8.33\text{ ms}^{-1}$$

As the two speeds v_1 and v_2 are positive, they move in the same direction with the velocities, 3.33 m s^{-1} and 8.33 m s^{-1} respectively.

4.4.3 Perfect inelastic collision

In a perfectly inelastic or completely inelastic collision, the objects stick together permanently after collision such that they move with common velocity. Let the two bodies with masses m_1 and m_2 move with initial velocities u_1 and u_2 respectively before collision. After perfect inelastic collision both the objects move together with a common velocity v as shown in Figure (4.17).

Since, the linear momentum is conserved during collisions,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\,v$$

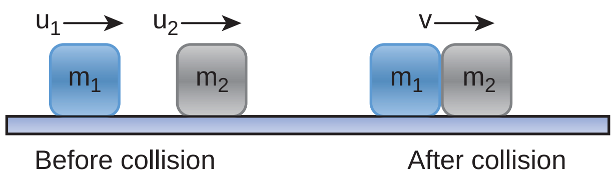


Figure 4.17 Perfect inelastic collision in one dimension

	Velocity		Linear momentum	
	Initial	Final	Initial	Final
Mass m_1	u_1	v	m_1u_1	m_1v
Mass m_2	u_2	v	m_2u_2	m_2v
Total			$m_1u_1 + m_2u_2$	(m_1+m_2) v

The common velocity can be computed by

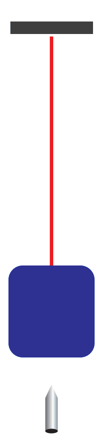
$$v = \frac{m_1u_1 + m_2u_2}{(m_1 + m_2)} \qquad (4.63)$$

EXAMPLE 4.21

A bullet of mass 50 g is fired from below into a suspended object of mass 450 g. The object rises through a height of 1.8 m with bullet remaining inside the object. Find the speed of the bullet. Take g = 10 ms^{-2}.

Solution

m_1 = 50 g = 0.05 kg; m_2 = 450 g = 0.45 kg



The speed of the bullet is u_1. The second body is at rest $(u_2 = 0)$. Let the common velocity of the bullet and the object after the bullet is embedded into the object is v.

$$v = \frac{m_1u_1 + m_2u_2}{(m_1 + m_2)}$$

$$v = \frac{0.05u_1 + (0.45 \times 0)}{(0.05 + 0.45)} = \frac{0.05}{0.50}u_1$$

The combined velocity is the initial velocity for the vertical upward motion of the combined bullet and the object. From second equation of motion,

$$v = \sqrt{2gh}$$

$$v = \sqrt{2 \times 10 \times 1.8} = \sqrt{36}$$

$$v = 6\,ms^{-1}$$

Substituting this in the above equation, the value of u_1 is

$$6 = \frac{0.05}{0.50}u_1 \quad \text{or} \quad u_1 = \frac{0.50}{0.05} \times 6 = 10 \times 6$$

$$u_1 = 60\,ms^{-1}$$

4.4.4 Loss of kinetic energy in perfect inelastic collision

In perfectly inelastic collision, the loss in kinetic energy during collision is transformed to another form of energy like sound, thermal, heat, light etc. Let KE_i be the total kinetic energy before collision and KE_f be the total kinetic energy after collision.

Total kinetic energy before collision,

$$KE_i = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \quad (4.64)$$

Total kinetic energy after collision,

$$KE_f = \frac{1}{2}(m_1 + m_2)v^2 \quad (4.65)$$

Then the loss of kinetic energy is
Loss of KE, $\Delta Q = KE_i - KE_f$

$$\Delta Q = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)v^2 \quad (4.66)$$

Substituting equation (4.63) in equation (4.66), and on simplifying (expand v by using the algebra $(a+b)^2 = a^2 + b^2 + 2ab$, we get

$$\text{Loss of KE, } \Delta Q = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(u_1 - u_2)^2 \quad (4.67)$$

4.4.5 Coefficient of restitution (e)

Suppose we drop a rubber ball and a plastic ball on the same floor. The rubber ball will bounce back higher than the plastic ball. This is because the loss of kinetic energy for an elastic ball is much lesser than the loss of kinetic energy for a plastic ball. The amount of kinetic energy after the collision of two bodies, in general, can be measured through a dimensionless number called the coefficient of restitution (COR).

It is defined as the ratio of velocity of separation (relative velocity) after collision to the velocity of approach (relative velocity) before collision, i.e.,

$$e = \frac{\text{velocity of separation (after collision)}}{\text{velocity of approach (before collision)}} = \frac{(v_2 - v_1)}{(u_1 - u_2)} \quad (4.68)$$

In an elastic collision, we have obtained the velocity of separation is equal to the velocity of approach i.e.,

$$(u_1 - u_2) = (v_2 - v_1) \rightarrow e = \frac{(v_2 - v_1)}{(u_1 - u_2)} = 1$$

This implies that, coefficient of restitution for an elastic collision, $e = 1$. Physically, it means that there is no loss of kinetic energy after the collision. So, the body bounces back with the same kinetic energy which is usually called as perfect elastic.

In any real collision problems, there will be some losses in kinetic energy due to collision, which means e is not always equal to unity. If the ball is perfectly plastic, it will never bounce back and therefore their separation of velocity is zero after the collision. Hence, the value of coefficient of restitution, $e = 0$.

In general, the coefficient of restitution for a material lies between $0 < e < 1$.

EXAMPLE 4.22

Show that the ratio of velocities of equal masses in an inelastic collision when one of the masses is stationary is $\frac{v_1}{v_2} = \frac{1-e}{1+e}$.

Solution

$$e = \frac{\text{velocity of separation (after collision)}}{\text{velocity of approach (before collision)}}$$

$$= \frac{(v_2 - v_1)}{(u_1 - u_2)} = \frac{(v_2 - v_1)}{(u_1 - 0)} = \frac{(v_2 - v_1)}{u_1}$$

$$\Rightarrow v_2 - v_1 = e u_1 \qquad (1)$$

From the law of conservation of linear momentum,

$$m u_1 = m v_1 + m v_2 \Rightarrow u_1 = v_1 + v_2 \quad (2)$$

Using the equation (2) for u_1 in (1), we get

$$v_2 - v_1 = e(v_1 + v_2)$$

On simplification, we get

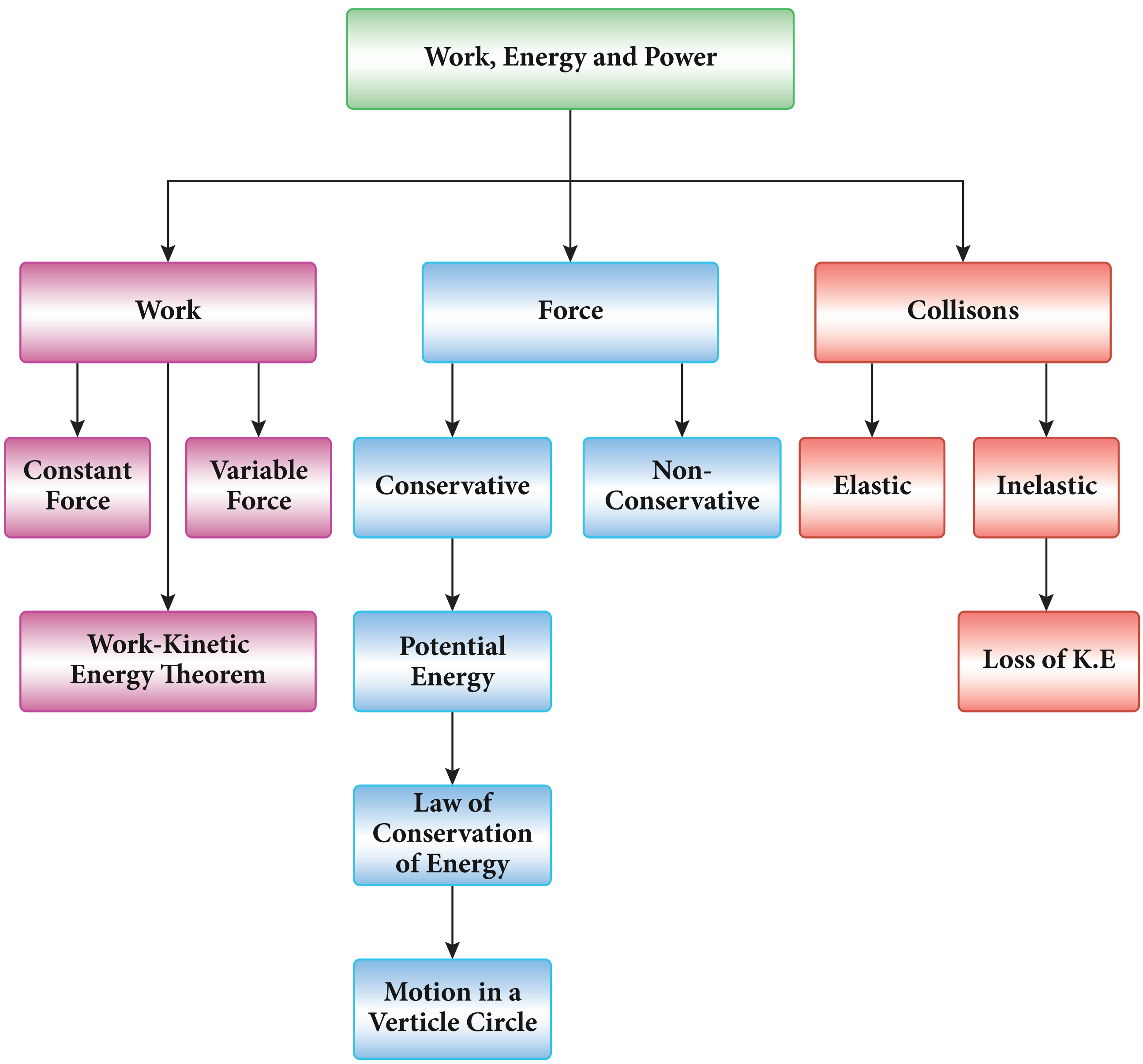
$$\frac{v_1}{v_2} = \frac{1-e}{1+e}$$

SUMMARY

- When a force $\vec{F}$ acting on an object displaces it by $d\vec{r}$, then the work done(W) by the force is $W = \vec{F}.d\vec{r} = Fdr\cos\theta$.
- The work done by the variable force is defined by $\int_i^f \vec{F}.d\vec{r}$
- Work-kinetic energy theorem: The work done by a force on the object is equal to the change in its kinetic energy.
- The kinetic energy can also be defined in terms of momentum which is given by $K.E = \frac{p^2}{2m}$.
- The potential energy at a point P is defined as the amount of work required to move the object from some reference point O to the point P with constant velocity. It is given by $U = \int_O^P \vec{F}_{ext}.d\vec{r}$. The reference point can be taken as zero potential energy.
- The gravitational potential energy at a height h is given by $U = mgh$. When the elongation or compression is x, the spring potential energy is given by $U = \frac{1}{2}kx^2$. Here k is spring constant.
- The work done by a conservative force around the closed path is zero and for a non-conservative force it is not zero.
- The gravitational force, spring force and Coulomb force are all conservative but frictional force is non-conservative.
- In the conservative force field, the total energy of the object is conserved.
- In the vertical circular motion, the minimum speed required by the mass to complete the circle is $\sqrt{5gr}$. Where r, is the radius of the circle.
- Power is defined as the rate of work done or energy delivered. It is equal to $P = \frac{W}{t} = \vec{F}.\vec{v}$
- The total linear momentum of the system is always conserved for both the elastic and inelastic collisions.
- The kinetic energy of the system is conserved in elastic collisions.
- The coefficient of restitution = $\frac{velocity\ of\ separation(after\ collision)}{velocity\ of\ approach(before\ collision)}$

UNIT-4(XI-Physics_Vol-1).indd 201 05-01-2022 22:20:44

CONCEPT MAP


Work, Energy and Power
Work
Force
Collisons
Constant Force
Variable Force
Conservative
Non-Conservative
Elastic
Inelastic
Work-Kinetic Energy Theorem
Potential Energy
Loss of K.E
Law of Conservation of Energy
Motion in a Verticle Circle

EVALUATION

I. Multiple Choice Questions

1. A uniform force of $(2\hat{i}+\hat{j})$ N acts on a particle of mass 1 kg. The particle displaces from position $(3\hat{j}+\hat{k})$ m to $(5\hat{i}+3\hat{j})$ m. The work done by the force on the particle is

   (AIPMT model 2013)

   (a) 9 J (b) 6 J
   (c) 10 J (d) 12 J

2. A ball of mass 1 kg and another of mass 2 kg are dropped from a tall building whose height is 80 m. After, a fall of 40 m each towards Earth, their respective kinetic energies will be in the ratio of

   (AIPMT model 2004)

   (a) $\sqrt{2}:1$ (b) $1:\sqrt{2}$
   (c) 2 : 1 (d) 1 : 2

3. A body of mass 1 kg is thrown upwards with a velocity 20 m s^{-1}. It momentarily comes to rest after attaining a height of 18 m. How much energy is lost due to air friction?.

   (Take $g=10 m s^{-2}$) (AIPMT 2009)

   (a) 20 J (b) 30 J
   (c) 40 J (d) 10 J

4. An engine pumps water continuously through a hose. Water leaves the hose with a velocity v and m is the mass per unit length of the water of the jet. What is the rate at which kinetic energy is imparted to water ?.

   (AIPMT 2009)

   (a) $\frac{1}{2}mv^3$ (b) mv^3
   (c) $\frac{3}{2}mv^2$ (d) $\frac{5}{2}mv^2$

5. A body of mass 4 m is lying in xy-plane at rest. It suddenly explodes into three pieces. Two pieces each of mass m move perpendicular to each other with equal speed v. The total kinetic energy generated due to explosion is

   (AIPMT 2014)

   (a) mv^2 (b) $\frac{3}{2}mv^2$
   (c) $2mv^2$ (d) $4mv^2$

6. The potential energy of a system increases, if work is done

   (a) by the system against a conservative force
   (b) by the system against a non-conservative force
   (c) upon the system by a conservative force
   (d) upon the system by a non-conservative force

7. What is the minimum velocity with which a body of mass m must enter a vertical loop of radius R so that it can complete the loop?.

   (a) $\sqrt{2gR}$ (b) $\sqrt{3gR}$
   (c) $\sqrt{5gR}$ (d) $\sqrt{gR}$

8. The work done by the conservative force for a closed path is

   (a) always negative
   (b) zero
   (c) always positive
   (d) not defined

9. If the linear momentum of the object is increased by 0.1%, then the kinetic energy is increased by

(a) 0.1 % (b) 0.2%

(c) 0.4% (d) 0.01%

10. If the potential energy of the particle is $\alpha - \frac{\beta}{2}x^2$, then force experienced by the particle is

(a) $F = \frac{\beta}{2}x^2$ (b) $F = \beta x$

(c) $F = -\beta x$ (d) $F = -\frac{\beta}{2}x^2$

11. A wind-powered generator converts wind energy into electric energy. Assume that the generator converts a fixed fraction of the wind energy intercepted by its blades into electrical energy. For wind speed v, the electrical power output will be proportional to

(a) v (b) v^2

(c) v^3 (d) v^4

12. Two equal masses m_1 and m_2 are moving along the same straight line with velocities $5\,ms^{-1}$ and $-9\,ms^{-1}$ respectively. If the collision is elastic, then calculate the velocities after the collision of m_1 and m_2, respectively

(a) $-4\,ms^{-1}$ and $10\,ms^{-1}$

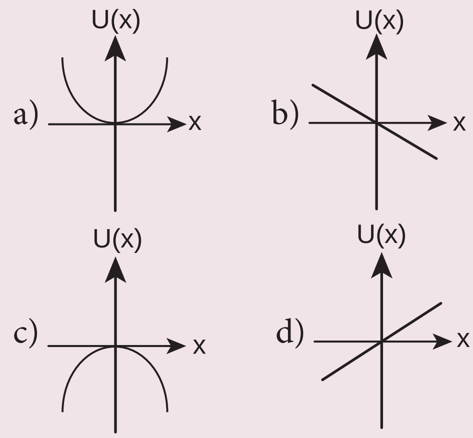
(b) $10\,ms^{-1}$ and $0\,ms^{-1}$

(c) $-9\,ms^{-1}$ and $5\,ms^{-1}$

(d) $5\,ms^{-1}$ and $1\,ms^{-1}$

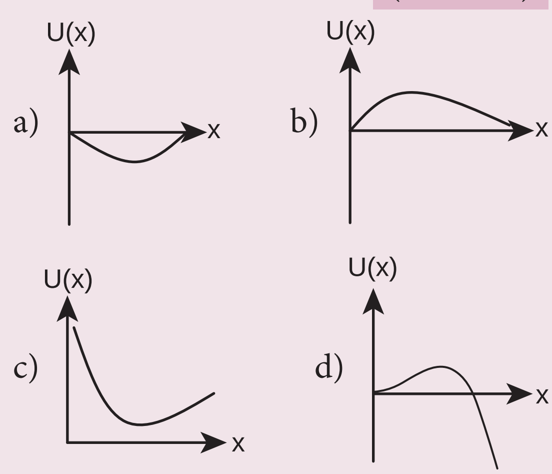
13. A particle is placed at the origin and a force $F = kx$ is acting on it (where k is a positive constant). If $U(0) = 0$, the graph of $U(x)$ versus x will be (where U is the potential energy function)

(IIT 2004)



14. A particle which is constrained to move along x-axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin as $F(x) = -kx + ax^3$. Here, k and a are positive constants. For $x \geq 0$, the functional form of the potential energy $U(x)$ of the particle is

(IIT 2002)



15. A spring of force constant k is cut into two pieces such that one piece is double the length of the other. Then, the long piece will have a force constant of

(a) $\frac{2}{3}k$ (b) $\frac{3}{2}k$

(c) $3k$ (d) $6k$

Answers

1) c	**2)** d	**3)** a	**4)** a	**5)** b
6) a	**7)** c	**8)** b	**9)** b	**10)** b
11) c	**12)** c	**13)** c	**14)** d	**15)** b

II. Short Answer Questions

1. Explain how the definition of work in physics is different from general perception.
2. Write the various types of potential energy. Explain the formulae.
3. Write the differences between conservative and Non-conservative forces. Give two examples each.
4. Explain the characteristics of elastic and inelastic collision.
5. Define the following
   a) Coefficient of restitution
   b) Power
   c) Law of conservation of energy
   d) loss of kinetic energy in inelastic collision.

III. Long Answer Questions

1. Explain with graphs the difference between work done by a constant force and by a variable force.
2. State and explain work energy principle. Mention any three examples for it.
3. Arrive at an expression for power and velocity. Give some examples for the same.
4. Arrive at an expression for elastic collision in one dimension and discuss various cases.
5. What is inelastic collision? In which way it is different from elastic collision. Mention few examples in day to day life for inelastic collision.

IV. Numerical Problems

1. Calculate the work done by a force of 30 N in lifting a load of 2kg to a height of 10m ($g = 10\ m\ s^{-2}$)

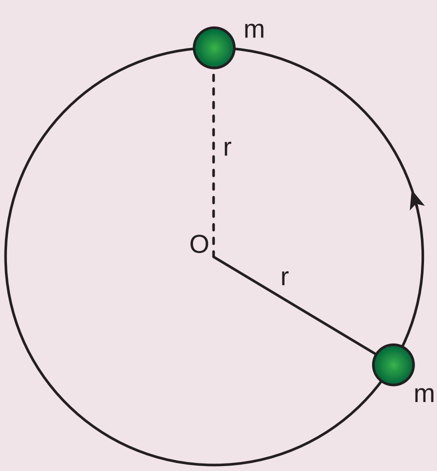
   Ans: 300J

2. A ball with a velocity of 5 ms^{-1} impinges at angle of 60˚ with the vertical on a smooth horizontal plane. If the coefficient of restitution is 0.5, find the velocity and direction after the impact.

   Ans: $v = 4.51\ m\ s^{-1}$

3. A bob of mass m is attached to one end of the rod of negligible mass and length r, the other end of which is pivoted freely at a fixed centre O as shown in the figure. What initial speed must be given to the object to reach the top of the circle? (Hint: Use law of conservation of energy). Is this speed less or greater than speed obtained in the section 4.2.9?

   Ans: $v = \sqrt{4gr}\ m\ s^{-1}$



4. Two different unknown masses A and B collide. A is initially at rest when B has a speed v. After collision B has a speed v/2 and moves at right angles to its original

direction of motion. Find the direction in which A moves after collision.

Ans: $\theta = 26° \, 33'$

5. A bullet of mass 20 g strikes a pendulum of mass 5 kg. The centre of mass of pendulum rises a vertical distance of 10 cm. If the bullet gets embedded into the pendulum, calculate its initial speed.

Ans: $v = 351.4 ms^{-1}$

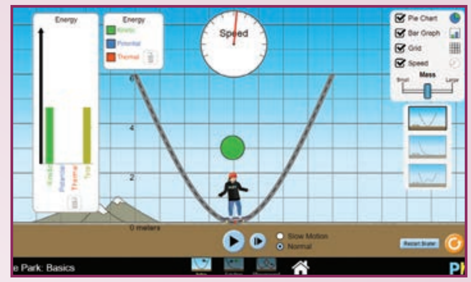
BOOKS FOR REFERENCE

1. Charles Kittel, Walter Knight, Malvin Ruderman, Carl Helmholtz and Moyer, *Mechanics,* 2nd edition, Mc Graw Hill Pvt Ltd,
2. A.P.French, *Newtonian Mechanics,* Viva-Norton Student edition
3. Somnath Datta, *Mechanics,* Pearson Publication
4. H.C.Verma, *Concepts of physics* volume 1 and Volume 2, Bharati Bhawan Publishers
5. Serway and Jewett, *Physics for scientist and Engineers with modern physics*, Brook/Coole publishers, Eighth edition
6. Paul Tipler and Gene Mosca, *Physics for scientist and engineers with modern physics*, Sixth edition, W.H. freeman and Company.

ICT CORNER

Energy Skate

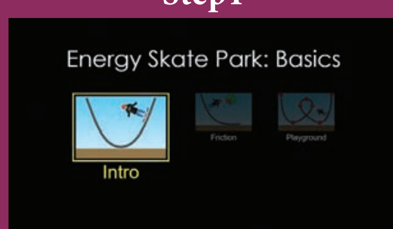
Through this activity you will understand about potential energy and kinetic energy



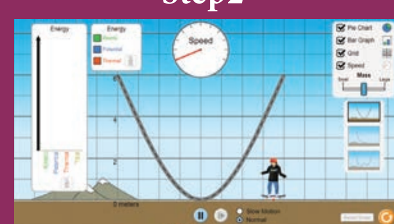
STEPS:

- Open the browser and type the given URL to open the PhET simulation on work power and energy. Click OK to open the activity.
- Select the height to observe the change in the kinetic energy, potential energy.
- Also observe the change by altering the mass.
- You can also create your own optional friction or playground to observe the change in potential energy and kinetic energy.

Step1

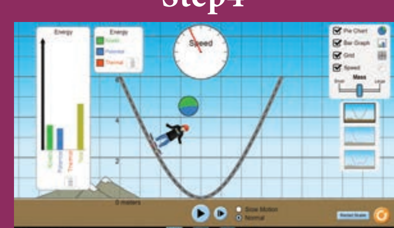


Step2



Step3

Step4



PhET simulation's URL:

https://phet.colorado.edu/en/simulation/energy-skate-park

*Pictures are indicative only.

*If browser requires, allow **Flash Player** or **Java Script** to load the page.

UNIT 5

MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES

"In nature, we have to deal not with material points but with material bodies ... – Max Planck

LEARNING OBJECTIVES

In this unit, the student is exposed to

- relevance of the centre of mass in various systems of particles
- torque and angular momentum in rotational motion
- types of equilibria with appropriate examples
- moment of inertia of different rigid bodies
- dynamics of rotation of rigid bodies
- distinguishing translational motion from rotational motion
- rolling motion, slipping and sliding motions.

5.1 INTRODUCTION

Most of the objects that we come across in our day to day life consist of large number of particles. In the previous Units, we studied the motion of bodies without considering their size and shape. So far we have treated even the bulk bodies as only point objects. In this section, we will give importance to the size and shape of the bodies. These bodies are actually made up of a large number of particles. When such a body moves, we consider it as the motion of collection of particles as a whole. We define the concept of centre of mass to deal with such a system of particles.

The forces acting on these bulk bodies are classified into *internal* and *external forces*. Internal forces are the forces acting among the particles within a system that constitute the body. External forces are the forces acting on the particles of a system from outside. In this unit, we deal with such system of particles which make different rigid bodies. *A rigid body is the one which maintains its definite and fixed shape even when an external force acts on it*. This means that, the interatomic distances do not change in a rigid body when an external force is applied. However, in real life situation, we have bodies which are not ideally rigid, because the shape and size of the body change when forces act on them. For the rigid bodies we study here, we assume that such deformations are negligible. The deformations produced on non-rigid bodies are studied separately in Unit 7 under elasticity of solids.

5.1.1 CENTRE OF MASS

When a rigid body moves, all particles that constitute the body need not take the same path. Depending on the type of motion, different particles of the body may take different paths. For example, when a wheel rolls on a surface, the path of the centre point of the wheel and the paths of other points of the wheel are different. In this Unit, we study about the translation, rotation and the combination of these motions of rigid bodies in detail.

5.1.2 Centre of Mass of a Rigid Body

When a bulk object (say a bat) is thrown at an angle in air as shown in Figure 5.1; do all the points of the body take a parabolic path? Actually, only one point takes the parabolic path and all the other points take different paths.

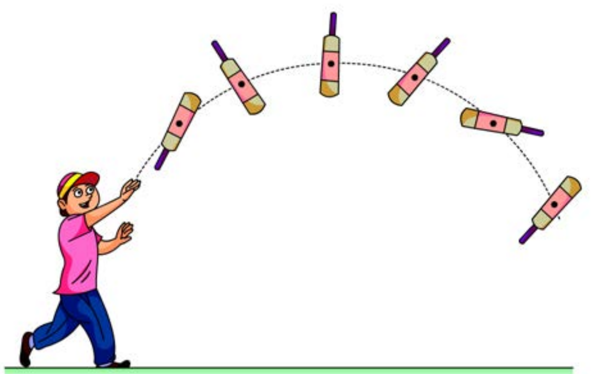


Figure 5.1 Centre of mass tracing the path of a parabola

The one point that takes the parabolic path is a very special point called *centre of mass* (CM) of the body. Its motion is like the motion of a single point that is thrown. *The centre of mass of a body is defined as a point where the entire mass of the body appears to be concentrated*. Therefore, this point can represent the entire body.

For bodies of regular shape and uniform mass distribution, the centre of mass is at the geometric centre of the body. As examples, for a circle and sphere, the centre of mass is at their centres; for square and rectangle, at the point their diagonals meet; for cube and cuboid, it is at the point where their body diagonals meet. For other bodies, the centre of mass has to be determined using some methods. The centre of mass could be well within the body and in some cases outside the body as well.

5.1.3 Centre of Mass for Distributed Point Masses

A point mass is a hypothetical point particle which has nonzero mass and no size or shape. To find the centre of mass for a collection of n point masses, say, $m_1, m_2, m_3 \ldots m_n$ we have to first choose an origin and an appropriate coordinate system as shown in Figure 5.2. Let, $x_1, x_2, x_3 \ldots x_n$ be the X-coordinates of the positions of these point masses in the X direction from the origin.

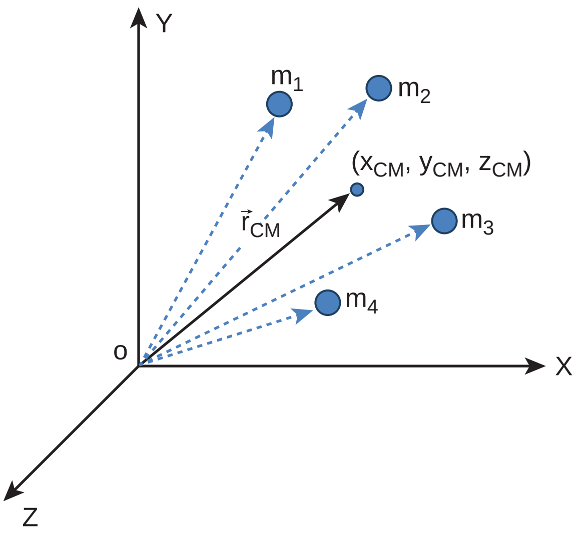


Figure 5.2 Centre of mass for distributed point masses

The equation for the x coordinate of the centre of mass is,

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i}$$

where, $\sum m_i$ is the total mass M of all the particles, $\left(\sum m_i = M\right)$. Hence,

$$x_{CM} = \frac{\sum m_i x_i}{M} \quad (5.1)$$

Similarly, we can also find y and z coordinates of the centre of mass for these distributed point masses as indicated in Figure (5.2).

$$y_{CM} = \frac{\sum m_i y_i}{M} \quad (5.2)$$

$$z_{CM} = \frac{\sum m_i z_i}{M} \quad (5.3)$$

Hence, the position of centre of mass of these point masses in a Cartesian coordinate system is (x_{CM}, y_{CM}, z_{CM}). In general, the position of centre of mass can be written in a vector form as,

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \quad (5.4)$$

where, $\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$ is the position vector of the centre of mass and $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ is the position vector of the distributed point mass; where, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along X, Y and Z-axes respectively.

5.1.4 Centre of Mass of Two Point Masses

With the equations for centre of mass, let us find the centre of mass of two point masses m_1 and m_2, which are at positions x_1 and x_2 respectively on the X-axis. For this case, we can express the position of centre of mass in the following three ways based on the choice of the coordinate system.

(i) When the masses are on positive X-axis:
The origin is taken arbitrarily so that the masses m_1 and m_2 are at positions x_1 and x_2 on the positive X-axis as shown in Figure 5.3(a). The centre of mass will also be on the positive X-axis at x_{CM} as given by the equation,

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

(ii) When the origin coincides with any one of the masses:
The calculation could be minimised if the origin of the coordinate system is made to coincide with any one of the masses as shown in Figure 5.3(b). When the origin coincides with the point mass m_1, its position x_1 is zero, (i.e. $x_1 = 0$). Then,

$$x_{CM} = \frac{m_1(0) + m_2 x_2}{m_1 + m_2}$$

The equation further simplifies as,

$$x_{CM} = \frac{m_2 x_2}{m_1 + m_2}$$

(iii) When the origin coincides with the centre of mass itself:
If the origin of the coordinate system is made to coincide with the centre of mass,

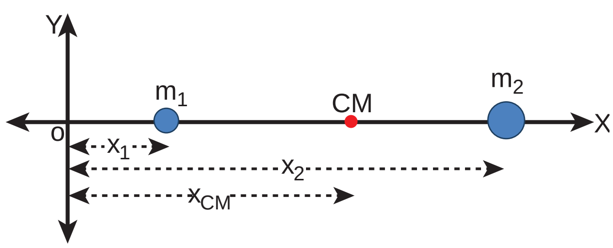
then, $x_{CM} = 0$ and the mass m_1 is found to be on the negative X-axis as shown in Figure 5.3(c). Hence, its position x_1 is negative, (i.e. $-x_1$).

$$0 = \frac{m_1(-x_1) + m_2x_2}{m_1 + m_2}$$

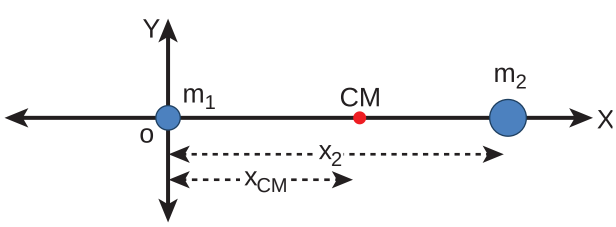
$$0 = m_1(-x_1) + m_2x_2$$

$$m_1x_1 = m_2x_2$$

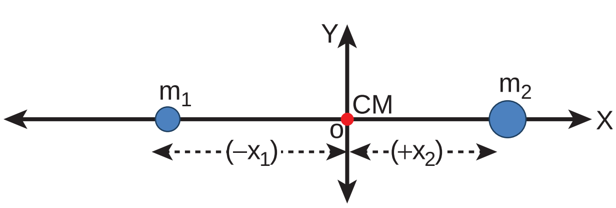
The equation given above is known as *principle of moments.* We will learn more about this in Section 5.3.3.



(a) When the masses are on positive X axis



(b) When the origin coincides with any one of the masses



(c) When the origin coincides with the center of mass itself

Figure 5.3 Centre of mass of two point masses determined by shifting the origin

EXAMPLE 5.1

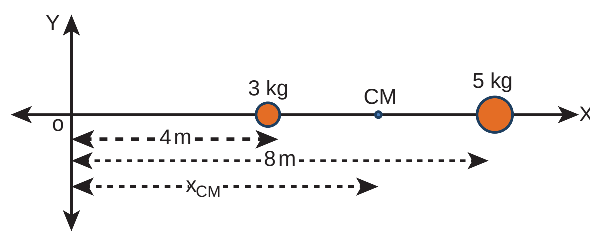
Two point masses 3 kg and 5 kg are at 4 m and 8 m from the origin on X-axis. Locate the position of centre of mass of the two point masses (i) from the origin and (ii) from 3 kg mass.

Solution

Let us take, $m_1 = 3$ kg and $m_2 = 5$ kg

(i) To find centre of mass from the origin:

The point masses are at positions, $x_1 = 4$ m, $x_2 = 8$ m from the origin along X axis.



The centre of mass x_{CM} can be obtained using equation 5.4.

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

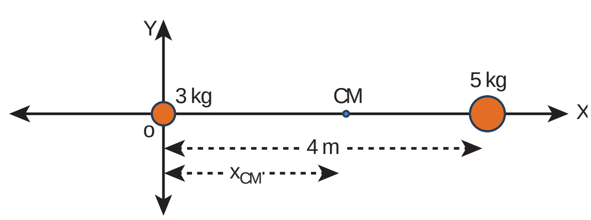
$$x_{CM} = \frac{(3 \times 4) + (5 \times 8)}{3 + 5}$$

$$x_{CM} = \frac{12 + 40}{8} = \frac{52}{8} = 6.5\,\text{m}$$

The centre of mass is located 6.5 m from the origin on X-axis.

(ii) To find the centre of mass from 3 kg mass:

The origin is shifted to 3 kg mass along X-axis. The position of 3 kg point mass is zero ($x_1 = 0$) and the position of 5 kg point mass is 4 m from the shifted origin ($x_2 = 4$ m).



UNIT-5(XI-Physics_Vol-1).indd 211 06-01-2022 17:46:20

$$x_{CM} = \frac{(3\times 0)+(5\times 4)}{3+5}$$

$$x_{CM} = \frac{0+20}{8} = \frac{20}{8} = 2.5\,m$$

The centre of mass is located 2.5 m from 3 kg point mass, (and 1.5 m from the 5 kg point mass) on X-axis.

- This result shows that the centre of mass is located closer to larger mass.
- If the origin is shifted to the centre of mass, then the principle of moments holds good. $m_1x_1 = m_2x_2$; $3\times 2.5 = 5\times 1.5$; $7.5 = 7.5$

When we compare case (i) with case (ii), the $x_{CM} = 2.5\,m$ from 3 kg mass could also be obtained by subtracting 4 m (the position of 3 kg mass) from 6.5 m, where the centre of mass was located in case (i)

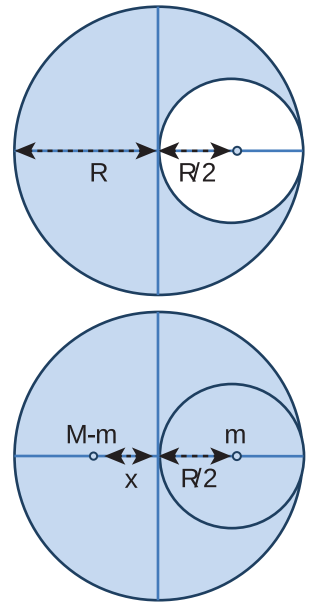
EXAMPLE 5.2

From a uniform disc of radius R, a small disc of radius $\frac{R}{2}$ is cut and removed as shown in the diagram. Find the centre of mass of the remaining portion of the disc.

Solution

Let us consider the mass of the uncut full disc be M. Its centre of mass would be at the geometric centre of the disc on which the origin coincides.

Let the mass of the small disc cut and removed be m and its centre of mass is at a position $\frac{R}{2}$ to the right of the origin as shown in the figure.



Hence, the remaining portion of the disc should have its centre of mass to the left of the origin; say, at a distance x. We can write from the principle of moments,

$$(M-m)x = (m)\frac{R}{2}$$

$$x = \left(\frac{m}{(M-m)}\right)\frac{R}{2}$$

If σ is the surface mass density (i.e. mass per unit surface area), $\sigma = \frac{M}{\pi R^2}$; then, the mass m of small disc is,

$$m = \text{surface mass density} \times \text{surface area}$$

$$m = \sigma \times \pi\left(\frac{R}{2}\right)^2$$

$$m = \left(\frac{M}{\pi R^2}\right)\pi\left(\frac{R}{2}\right)^2 = \frac{M}{\pi R^2}\pi\frac{R^2}{4} = \frac{M}{4}$$

substituting m in the expression for x

$$x = \frac{\frac{M}{4}}{\left(M-\frac{M}{4}\right)} \times \frac{R}{2} = \frac{\frac{M}{4}}{\left(\frac{3M}{4}\right)} \times \frac{R}{2}$$

$$x = \frac{R}{6}$$

The centre of mass of the remaining portion is at a distance $\frac{R}{6}$ to the left from the centre of the disc.

- If, the small disc is removed concentrically from the large disc, what will be the position of the centre of mass of the remaining portion of disc?

EXAMPLE 5.3

The position vectors of two point masses 10 kg and 5 kg are $(-3\hat{i}+2\hat{j}+4\hat{k})$ m and $(3\hat{i}+6\hat{j}+5\hat{k})$ m respectively. Locate the position of centre of mass.

Solution

$$m_1 = 10\,kg$$

$$m_2 = 5\,kg$$

$$\vec{r}_1 = (-3\hat{i}+2\hat{j}+4\hat{k})m$$

$$\vec{r}_2 = (3\hat{i}+6\hat{j}+5\hat{k})m$$

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$\therefore \vec{r} = \frac{10(-3\hat{i}+2\hat{j}+4\hat{k})+5(3\hat{i}+6\hat{j}+5\hat{k})}{10+5}$$

$$= \frac{-30\hat{i}+20\hat{j}+40\hat{k}+15\hat{i}+30\hat{j}+25\hat{k}}{15}$$

$$= \frac{-15\hat{i}+50\hat{j}+65\hat{k}}{15}$$

$$\vec{r} = \left(-\hat{i}+\frac{10}{3}\hat{j}+\frac{13}{3}\hat{k}\right) m$$

The centre of mass is located at position $\vec{r}$.

5.1.5 Centre of mass for uniform distribution of mass

If the mass is uniformly distributed in a bulk object, then a small mass (Δm) of the body can be treated as a point mass and the summations can be done to obtain the expressions for the coordinates of centre of mass.

$$x_{CM} = \frac{\sum(\Delta m_i)x_i}{\sum \Delta m_i}$$

$$y_{CM} = \frac{\sum(\Delta m_i)y_i}{\sum \Delta m_i} \qquad (5.5)$$

$$z_{CM} = \frac{\sum(\Delta m_i)z_i}{\sum \Delta m_i}$$

On the other hand, if the small mass taken is infinitesimally* small (dm) then, the summations can be replaced by integrations as given below.

$$x_{cm} = \frac{\int x\,dm}{\int dm}$$

$$y_{cm} = \frac{\int y\,dm}{\int dm} \qquad (5.6)$$

$$z_{cm} = \frac{\int z\,dm}{\int dm}$$

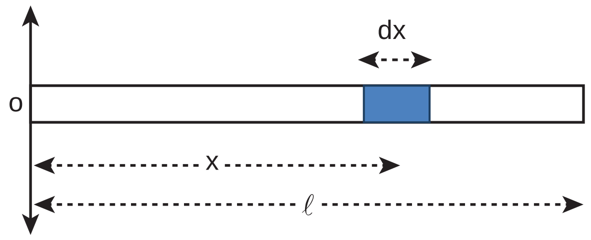
EXAMPLE 5.4

Locate the centre of mass of a uniform rod of mass M and length ℓ.

Solution

Consider a uniform rod of mass M and length ℓ whose one end coincides with the origin as shown in Figure. The rod is kept along the x axis. To find the centre of mass

* Infinitesimal quantity is an extremely small quantity.



of this rod, we choose an infinitesimally small mass dm of elemental length dx at a distance x from the origin.

λ is the linear mass density (i.e. mass per unit length) of the rod. $\lambda = \frac{M}{\ell}$

The mass of small element (dm) is, $dm = \frac{M}{\ell} dx$

Now, we can write the centre of mass equation for this mass distribution as,

$$x_{CM} = \frac{\int x\,dm}{\int dm}$$

$$x_{CM} = \frac{\int_{o}^{\ell} x\left(\frac{M}{\ell}dx\right)}{M} = \frac{1}{\ell}\int_{o}^{\ell} x\,dx$$

$$= \frac{1}{\ell}\left[\frac{x^2}{2}\right]_0^{l} = \frac{1}{\ell}\left(\frac{\ell^2}{2}\right)$$

$$x_{CM} = \frac{\ell}{2}$$

As the position $\frac{\ell}{2}$ is the geometric centre of the rod, it is concluded that the centre of mass of the uniform rod is located at its geometric centre itself.

5.1.6 Motion of Centre of Mass

When a rigid body moves, its centre of mass will also move along with the body. For kinematic quantities like velocity (v_{CM}) and acceleration (a_{CM}) of the centre of mass, we can differentiate the expression for position of centre of mass with respect to time once and twice respectively. For simplicity, let us take the motion along X direction only.

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt} = \frac{\sum m_i\left(\frac{d\vec{x}_i}{dt}\right)}{\sum m_i}$$

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i} \qquad (5.7)$$

$$\vec{a}_{CM} = \frac{d}{dt}\left(\frac{d\vec{x}_{CM}}{dt}\right) = \left(\frac{d\vec{v}_{CM}}{dt}\right) = \frac{\sum m_i\left(\frac{d\vec{v}_i}{dt}\right)}{\sum m_i}$$

$$\vec{a}_{CM} = \frac{\sum m_i \vec{a}_i}{\sum m_i} \qquad (5.8)$$

In the absence of external force, i.e. $\vec{F}_{ext} = 0$, the individual rigid bodies of a system can move or shift only due to the internal forces. This will not affect the position of the centre of mass. This means that the centre of mass will be in a state of rest or uniform motion. Hence, $\vec{v}_{CM}$ will be zero when centre of mass is at rest and constant when centre of mass has uniform motion $(\vec{v}_{CM} = 0 \text{ or } \vec{v}_{CM} = \text{constant})$. There will be no acceleration of centre of mass, $(\vec{a}_{CM} = 0)$.

From equation (5.7) and 5.8,

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i} = 0 \text{ (or) } \vec{v}_{CM} = \text{constant}$$

It implies

$$\vec{a}_{CM} = \frac{\sum m_i \vec{a}_i}{\sum m_i} = 0$$

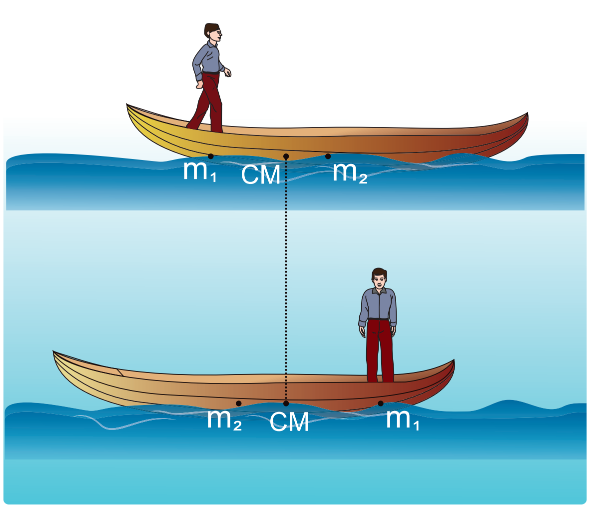
Here, the individual particles may still move with their respective velocities and accelerations due to internal forces.

In the presence of external force, (i.e. $\vec{F}_{ext} \neq 0$), the centre of mass of the system will accelerate as given by the following equation.

$$\vec{F}_{ext} = \left(\sum m_i\right)\vec{a}_{CM}; \quad \vec{F}_{ext} = M\vec{a}_{CM}; \quad \vec{a}_{CM} = \frac{\vec{F}_{ext}}{M}$$

EXAMPLE 5.5

A man of mass 50 kg is standing at one end of a boat of mass 300 kg floating on still water. He walks towards the other end of the boat with a constant velocity of 2 m s^{-1} with respect to a stationary observer on land. What will be the velocity of the boat, (a) with respect to the stationary observer on land? (b) with respect to the man walking in the boat?



[Given: There is friction between the man and the boat and no friction between the boat and water.]

Solution

Mass of the man (m_1) is, m_1= 50 kg

Mass of the boat (m_2) is, m_2 = 300 kg

With respect to a stationary observer:

The man moves with a velocity, v_1 = 2 m s^{-1} and the boat moves with a velocity v_2 (which is to be found)

(i) To determine the velocity of the boat with respect to a stationary observer on land:

As there is no external force acting on the system, the man and boat move due to the friction, which is an internal force in the boat-man system. Hence, the velocity of the centre of mass is zero (v_{CM} = 0).

Using equation 5.7,

$$0 = \frac{\sum m_i v_i}{\sum m_i} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$0 = m_1 v_1 + m_2 v_2$$

$$-m_2 v_2 = m_1 v_1$$

$$v_2 = -\frac{m_1}{m_2} v_1$$

$$v_2 = -\frac{50}{300} \times 2 = -\frac{100}{300}$$

$$v_2 = -0.33 \text{ m s}^{-1}$$

The negative sign in the answer implies that the boat moves in a direction opposite to that of the walking man on the boat to a stationary observer on land.

(ii) To determine the velocity of the boat with respect to the walking man:

We can find the relative velocity as,

$$v_{21} = v_2 - v_1$$

where, v_{21} is the relative velocity of the boat with respect to the walking man.

$$v_{21} = (-0.33) - (2)$$
$$v_{21} = -2.33\,ms^{-1}$$

The negative sign in the answer implies that the boat appears to move in the opposite direction to the man walking in the boat.

- The magnitude of the relative velocity of the boat with respect to the walking man is greater than the magnitude of the relative velocity of the boat with respect to the stationary observer.
- The negative signs in the two answers indicate the opposite direction of the boat with respect to the stationary observer and the walking man on the boat.

Centre of mass in explosions:

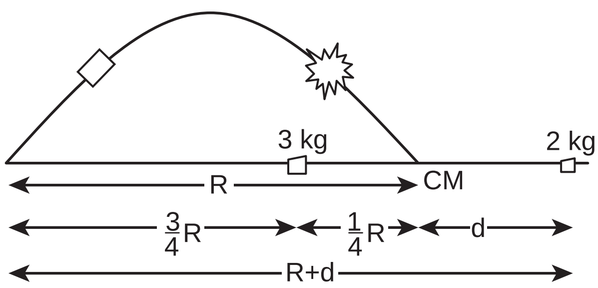
Many a times rigid bodies are broken in to fragments. If an explosion is caused by the internal forces in a body which is at rest or in motion, the state of the centre of mass is not affected. It continues to be in the same state of rest or motion. But, the kinematic quantities of the fragments get affected. If the explosion is caused by an external agency, then the kinematic quantities of the centre of mass as well as the fragments get affected.

EXAMPLE 5.6

A projectile of mass 5 kg, in its course of motion explodes on its own into two fragments. One fragment of mass 3 kg falls at three fourth of the range R of the projectile. Where will the other fragment fall?

Solution

It is an explosion of its own without any external influence. After the explosion, the centre of mass of the projectile will continue to complete the parabolic path even though the fragments are not following the same parabolic path. After the fragments have fallen on the ground, the centre of mass rests at a distance R (the range) from the point of projection as shown in the diagram.



If the origin is fixed to the final position of the centre of mass, the principle of moments holds good.

$$m_1x_1 = m_2x_2$$

where, m_1 = 3 kg, m_2 = 2 kg, $x_1 = \frac{1}{4}R$. The value of $x_2 = d$

$$3 \times \frac{1}{4}R = 2 \times d;$$
$$d = \frac{3}{8}R$$

The distance between the point of launching and the position of 2 kg mass is R+d.

$$R + d = R + \frac{3}{8}R = \frac{11}{8}R = 1.375R$$

The other fragment falls at a distance of 1.375R from the point of launching. (Here R is the range of the projectile.)

5.2 TORQUE AND ANGULAR MOMENTUM

When a net force acts on a body, it produces linear motion in the direction of the applied force. If the body is fixed to a point or an axis, such a force rotates the body depending on the point of application of the force on the body. This ability of the force to produce rotational motion in a body is called torque or moment of force. Examples for such motion are plenty in day to day life. To mention a few; the opening and closing of a door about the hinges and turning of a nut using a wrench.

4P500

The extent of the rotation depends on the magnitude of the force, its direction and the distance between the fixed point and the point of application. When torque produces rotational motion in a body, its angular momentum changes with respect to time. In this Section we will learn about the torque and its effect on rigid bodies.

5.2.1 Definition of Torque

Torque is defined as the moment of the external applied force about a point or axis of rotation. The expression for torque is,

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad (5.9)$$

where, $\vec{r}$ is the position vector of the point where the force $\vec{F}$ is acting on the body as shown in Figure 5.4.

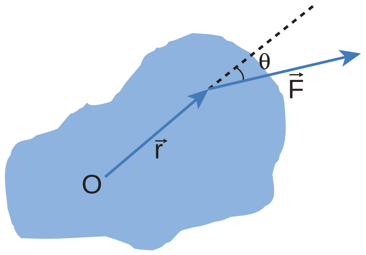


Figure 5.4 Torque on a rigid body

Here, the product of $\vec{r}$ and $\vec{F}$ is called the *vector product or cross product*. The vector product of two vectors results in another vector that is perpendicular to both the vectors (refer Section 2.5.2). Hence, torque $(\vec{\tau})$ is a vector quantity.

Torque has a magnitude $rF\sin\theta$ and direction perpendicular to $\vec{r}$ and $\vec{F}$. Its unit is N m.

$$\vec{\tau} = (rF\sin\theta)\hat{n} \qquad (5.10)$$

Here, θ is the angle between $\vec{r}$ and $\vec{F}$, and $\hat{n}$ is the unit vector in the direction of $\vec{\tau}$. Torque $(\vec{\tau})$ is sometimes called as a *pseudo vector* as it needs the other two vectors $\vec{r}$ and $\vec{F}$ for its existence.

The direction of torque is found using right hand rule. This rule says that if fingers of right hand are kept along the position vector with palm facing the direction of the force and when the fingers are curled the thumb points to the direction of the torque. This is shown in Figure 5.5.

The direction of torque helps us to find the type of rotation caused by the torque. For example, if the direction of torque is out

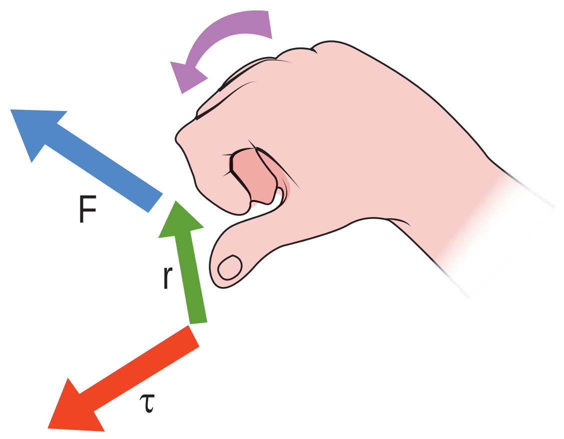


Figure 5.5 Direction of torque using right hand rule

of the paper, then the rotation produced by the torque is anticlockwise. On the other hand, if the direction of the torque is into the paper, then the rotation is clockwise as shown in Figure 5.6.

In many cases, the direction and magnitude of the torque are found separately. For direction, we use the vector rule or right hand rule. For magnitude, we use scalar form as,

$$\tau = r\ F\ \sin\theta \qquad (5.11)$$

The expression for the magnitude of torque can be written in two different ways by associating $\sin\theta$ either with r or F in the following manner.

$$\tau = r(F\ \sin\theta) = r \times (F\perp) \qquad (5.12)$$

$$\tau = (r\ \sin\theta)F = (r\perp)\times F \qquad (5.13)$$

Here, $(F\ \sin\theta)$ is the component of $\vec{F}$ perpendicular to $\vec{r}$. Similarly, $(r\ \sin\theta)$ is the component of $\vec{r}$ perpendicular to $\vec{F}$. The two cases are shown in Figure 5.7.

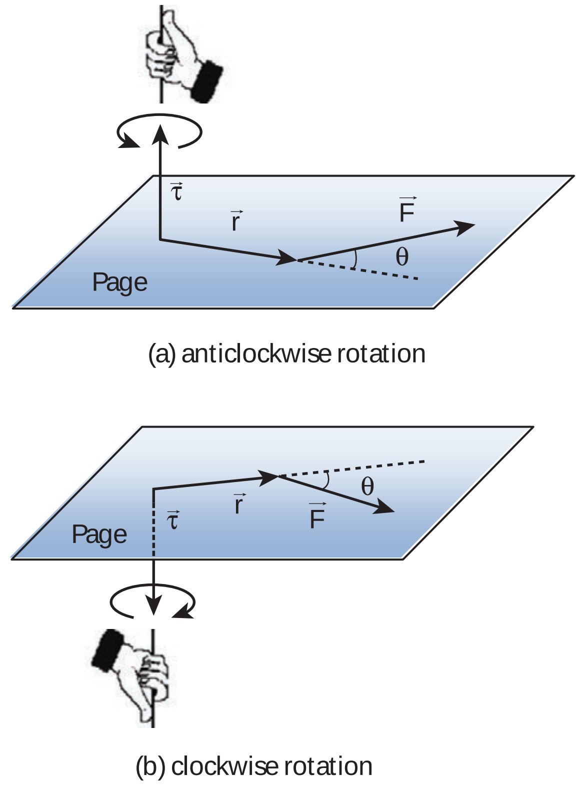


Figure 5.6 Direction of torque and the type of rotation

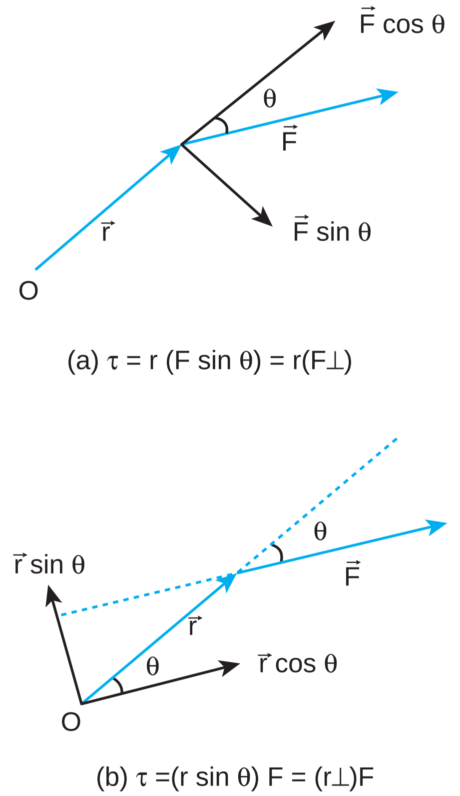


Figure 5.7 Two ways of calculating the torque.

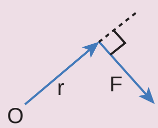
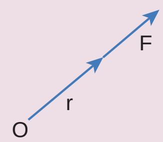
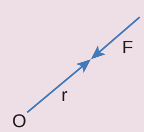
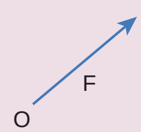
Based on the angle θ between $\vec{r}$ and $\vec{F}$, the torque takes different values.

The torque is maximum when, $\vec{r}$ and $\vec{F}$ are perpendicular to each other. That is when $\theta = 90^{\circ}$ and $\sin 90^{\circ} = 1$, Hence, $\tau_{max} = rF$.

The torque is zero when $\vec{r}$ and $\vec{F}$ are parallel or antiparallel. If parallel, then $\theta = 0^{\circ}$ and $\sin 0^{\circ} = 0$. If antiparallel, then $\theta = 180^{\circ}$ and $\sin 180^{\circ} = 0$. Hence, $\tau = 0$.

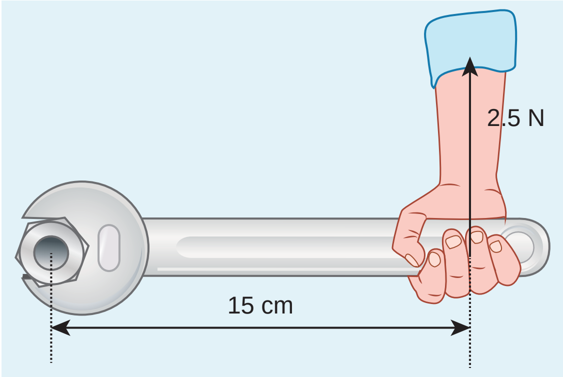
The torque is zero if the force acts at the reference point. i.e. as $\vec{r} = 0$, $\tau = 0$. The different cases discussed are shown in Table 5.1.

Table 5.1 The Value of τ for different cases.

 $\theta = 90^{\circ};\ \tau_{max} = rF$	 $\theta = 0^{\circ};\ \tau = 0$
 $\theta = 180^{\circ};\ \tau = 0$	 $r = 0;\ \tau = 0$

EXAMPLE 5.7

If the force applied is perpendicular to the handle of the spanner as shown in the diagram, find the (i) torque exerted by the force about the centre of the nut, (ii) direction of torque and (iii) type of rotation caused by the torque about the nut.

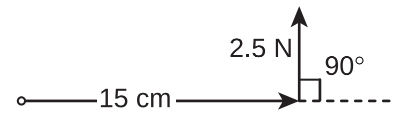


Solution

Arm length of the spanner, r = 15 cm $= 15\times10^{-2}$m

Force, F = 2.5 N

Angle between r and F, $\theta = 90^{\circ}$



(i) Torque, $\tau = rF\ \sin\theta$

$$\tau = 15\times10^{-2}\times2.5\times\sin(90^{\circ})$$

[here, sin 90° = 1]

$$\tau = 37.5\times10^{-2}\ \text{N m}$$

(ii) As per the right hand rule, the direction of torque is out of the page.

(iii) The type of rotation caused by the torque is anticlockwise.

EXAMPLE 5.8

A force of $(4\hat{i}-3\hat{j}+5\hat{k})$ N is applied at a point whose position vector is $(7\hat{i}+4\hat{j}-2\hat{k})$ m. Find the torque of force about the origin.

Solution

$$\vec{r} = 7\hat{i}+4\hat{j}-2\hat{k}$$

$$\vec{F} = 4\hat{i}-3\hat{j}+5\hat{k}$$

Torque, $\vec{\tau} = \vec{r}\times\vec{F}$

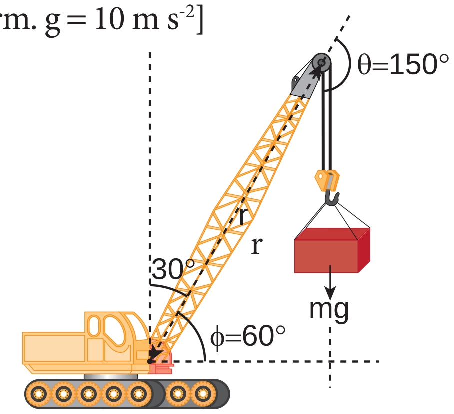
$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 4 & -2 \\ 4 & -3 & 5 \end{vmatrix}$$

$$\vec{\tau} = \hat{i}(20-6)-\hat{j}(35+8)+\hat{k}(-21-16)$$

$$\vec{\tau} = (14\hat{i}-43\hat{j}-37\hat{k})\ \text{N m}$$

EXAMPLE 5.9

A crane has an arm length of 20 m inclined at 30° with the vertical. It carries a container of mass of 2 ton suspended from the top end of the arm. Find the torque produced by the gravitational force on the container about the point where the arm is fixed to the crane. [Given: 1 ton = 1000 kg; neglect the weight of the arm. $g = 10\ m\ s^{-2}$]



Solution

Note In many problems, the angle θ between $\vec{r}$ and $\vec{F}$ will not be directly given. Thus, the students must get accustomed to identify and denote always the angle between the $\vec{r}$ and $\vec{F}$ as θ. The other angles in the arrangement may be denoted as α, β, ϕ etc.

The force F at the point of suspension is due to the weight of the hanging mass.

$$F = mg = 2 \times 1000 \times 10 = 20000\ N;$$
The arm length, $r = 20$ m

We can solve this problem by three different methods.

Method – I

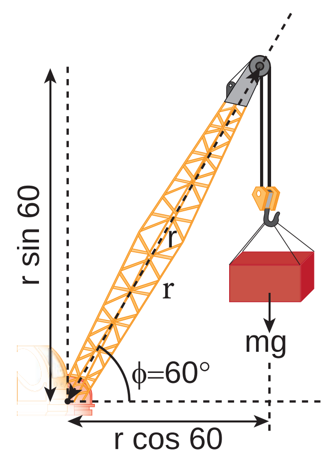
The angle (θ) between the arm length (r) and the force (F) is, θ = 150°

The torque (τ) about the fixed point of the arm is,

$$\tau = rF\ \sin\theta$$
$$\tau = 20 \times 20000 \times \sin(150^\circ)$$
$$= 400000 \times \sin(90^\circ + 60^\circ)$$
$$[\text{here, } \sin(90^\circ + \theta) = \cos\theta]$$
$$= 400000 \times \cos(60^\circ)$$
$$= 400000 \times \frac{1}{2} \quad \left[\cos 60^\circ = \frac{1}{2}\right]$$
$$= 200000\ \text{N m}$$
$$\tau = 2 \times 10^5\ \text{N m}$$

Method – II

Let us take the force and perpendicular distance from the point where the arm is fixed to the crane.

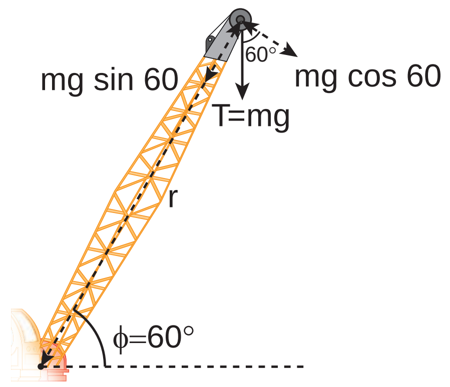


$$\tau = (r\perp)F$$
$$\tau = r\cos\phi\ mg$$
$$\tau = 20 \times \cos 60^\circ \times 20000$$
$$= 20 \times \frac{1}{2} \times 20000$$
$$= 200000\ \text{Nm}$$
$$\tau = 2 \times 10^5\ \text{Nm}$$

Method – III

Let us take the distance from the fixed point and perpendicular force.

UNIT-5(XI-Physics_Vol-1).indd 220 06-01-2022 17:46:39



$$\tau = r(F\perp)$$
$$\tau = r\,mg\ \cos\phi$$
$$\tau = 20\times 20000\times \cos 60^{\circ}$$
$$= 20\times 20000\times \frac{1}{2}$$
$$= 200000\ \mathrm{Nm}$$
$$\tau = 2\times 10^{5}\,\mathrm{Nm}$$

All the three methods, give the same answer.

5.2.2 Torque about an Axis

In the earlier sections, we have dealt with the torque about a point. In this section we will deal with the torque about an axis. Let us consider a rigid body capable of rotating about an axis AB as shown in Figure 5.8. Let the force F act at a point P on the rigid body. *The force F may not be on the plane ABP*. We can take the origin O at any random point on the axis AB.

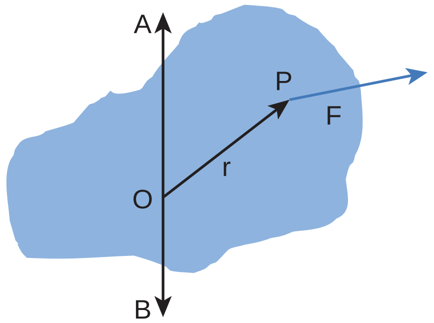


Figure 5.8 Torque about an axis

Tamil Nadu is known for creative and innovative traditional games played by children. One such very popular game is “silli” (சில்லி) or “sillukodu” (சில்லுக்கோடு). There is a rectangular area which is further partitioned as seen in the Figure. One has to hop through the rectangles. While doing so, children lean on one side, because of the reason that naturally the body takes this position to balance the gravitational force (mg) and normal force (N) acting on the body and to nullify the torque. Failing which, both these forces act along different lines leading to a net torque which makes one to fall.

The torque of the force $\vec{F}$ about O is, $\vec{\tau}=\vec{r}\times\vec{F}$. The component of the torque $\vec{\tau}$ along the axis is the torque about the axis. To find it, we should first find the vector $\vec{\tau}=\vec{r}\times\vec{F}$ and then find the angle ϕ between $\vec{\tau}$ and the axis AB. (Remember here, the force $\vec{F}$ is not on the plane ABP). The torque about the axis AB is the parallel component of the torque along the axis AB, which is $\left|\vec{r}\times\vec{F}\right|\cos\phi$. The torque perpendicular to the axis AB is $\left|\vec{r}\times\vec{F}\right|\sin\phi$.

The torque about the axis will rotate the object about the axis and the torque perpendicular to the axis will turn or tilt the axis of rotation itself. When both components exist simultaneously on a rigid body, the body will have a *precession*. One can witness the precessional motion in a spinning top when it is about to come to rest as shown in Figure 5.9.

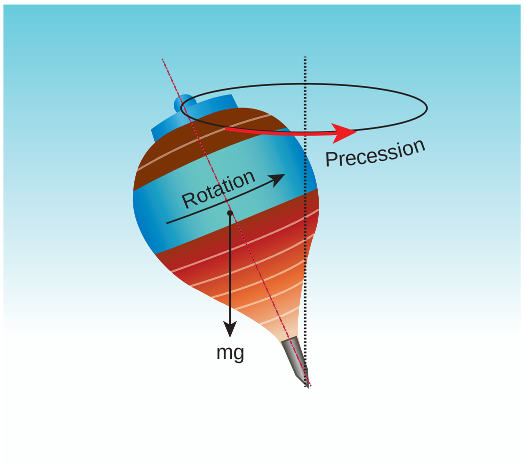


Figure 5.9 Precession of a spinning top

Study of precession is beyond the scope of the present course of study. Hence, it is assumed that there are constraints to cancel the effect of the perpendicular components of the torques, so that the fixed position of the axis is maintained. Therefore, perpendicular components of the torque need not be taken into account.

For the rest of the lesson, we consider rotation about only fixed axis. For this we shall,

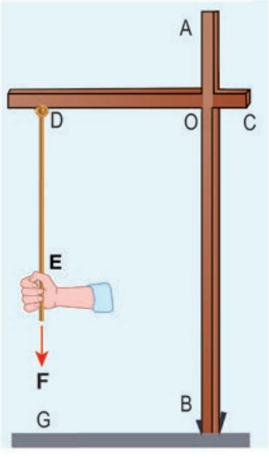
1. *Consider forces that lie only on planes perpendicular to the axis (without intersecting in the axis).*
2. *Consider position vectors that are only perpendicular to the axis.*

Note

- Forces parallel to the axis will give torques perpendicular to the axis of rotation and need not be taken into account.
- Forces that intersect (pass through) the axis cannot produce torque as r = 0.
- Position vectors along the axis will result in torques perpendicular to the axis and need not be taken into account.

EXAMPLE 5.10

Two mutually perpendicular beams AB, CD, are joined at O to form a structure which is fixed to the ground firmly as shown in the Figure. A string is tied to the point D and its free end E is pulled with a force $\vec{F}$. Find the magnitude and direction of the torque produced by the force,



(i) about the points E, D, O and B,

(ii) about the axes DE, CD, AB and BG.

Solution

(i) Torque about point E is zero. (as $\vec{F}$ passes through E).

Torque about point D is zero. (as $\vec{F}$ passes through D).

Torque about point O is $(\overrightarrow{OE}) \times \vec{F}$ which is perpendicular to axes AB and CD.

Torque about point B is $(\overrightarrow{BE}) \times \vec{F}$ which is perpendicular to axes AB and CD.

(ii) Torque about axis DE is zero (as $\vec{F}$ is parallel to DE).

Torque about axis CD is zero (as $\vec{F}$ intersects CD).

Torque about axis AB is zero (as $\vec{F}$ is parallel to AB).

Torque about axis BG is zero (as $\vec{F}$ intersects BG).

The torque of a force about an axis is independent of the choice of the origin as long as it is chosen on that axis itself. This can be shown as below.

Let O be the origin on the axis AB, which is the rotational axis of a rigid body. F is the force acting at the point P. Now, choose another point O' anywhere on the axis as shown in Figure 5.10.

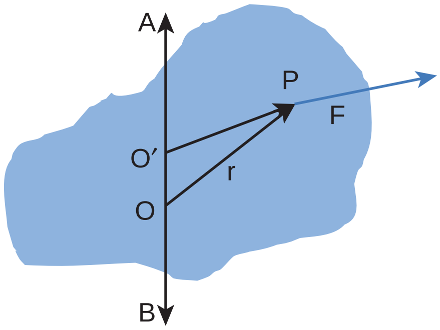


Figure 5.10 Torque about an axis is independent of origin

Identify the direction of torque in country press shown in picture (in Tamil, 'Marasekku' மரச்செக்கு)

The torque of F about O' is,

$$\overrightarrow{O'P} \times \vec{F} = \left(\overrightarrow{O'O} + \overrightarrow{OP}\right) \times \vec{F}$$
$$= \left(\overrightarrow{O'O} \times \vec{F}\right) + \left(\overrightarrow{OP} \times \vec{F}\right)$$

As $\overrightarrow{O'O} \times \vec{F}$ is perpendicular to $\overrightarrow{O'O}$, this term will not have a component along AB. Thus, the component of $\overrightarrow{O'P} \times \vec{F}$ is equal to that of $\overrightarrow{OP} \times \vec{F}$.

5.2.3 Torque and Angular Acceleration

Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about a fixed axis as shown in Figure 5.11. A tangential force $\vec{F}$ acting on the point mass produces the necessary torque for this rotation. This force $\vec{F}$ is perpendicular to the position vector $\vec{r}$ of the point mass.

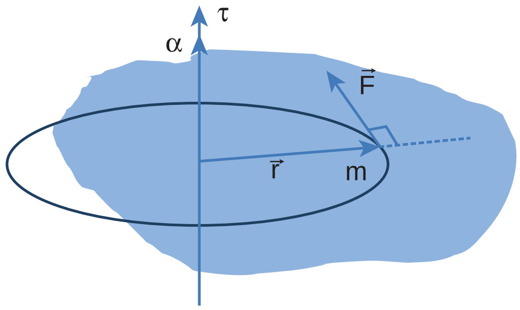


Figure 5.11 Torque and Angular acceleration

The torque produced by the force on the point mass m about the axis can be written as,

$$\tau = rF\sin 90° = rF \qquad [\because \sin 90° = 1]$$
$$\tau = r\,ma \qquad [\because (F = ma)]$$
$$\tau = r\,m\,r\alpha = mr^2\alpha \qquad [\because (a = r\alpha)]$$

$$\tau = \left(mr^2\right)\alpha \qquad (5.14)$$

Hence, the torque of the force acting on the point mass produces an angular acceleration (α) in the point mass about the axis of rotation.

In vector notation,

$$\vec{\tau} = \left(mr^2\right)\vec{\alpha} \qquad (5.15)$$

The directions of τ and α are along the axis of rotation. If the direction of τ is in the direction of α, it produces angular acceleration. On the other hand if, τ is opposite to α, angular deceleration or retardation is produced on the point mass.

The term mr^2 in equations 5.14 and 5.15 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $\left(I = \sum m_i r_i^2\right)$. Hence, torque for the rigid body can be written as,

$$\vec{\tau} = \left(\sum m_i r_i^2\right)\vec{\alpha} \qquad (5.16)$$
$$\vec{\tau} = I\vec{\alpha} \qquad (5.17)$$

We will learn more about the moment of inertia and its significance for bodies with different shapes in section 5.4.

5.2.4 Angular Momentum

The angular momentum in rotational motion is equivalent to linear momentum in translational motion. The angular momentum of a point mass is defined as the moment of its linear momentum. In other words, the angular momentum L of a point mass having a linear momentum p at a position r with respect to a point or axis is mathematically written as,

$$\vec{L} = \vec{r} \times \vec{p} \qquad (5.18)$$

The magnitude of angular momentum could be written as,

$$L = r\,p \sin\theta \qquad (5.19)$$

where, θ is the angle between $\vec{r}$ and $\vec{p}$. $\vec{L}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$. As we have written in the case of torque, here also we can associate $\sin\theta$ with either $\vec{r}$ or $\vec{p}$.

$$L = r\left(p\sin\theta\right) = r\left(p\perp\right) \qquad (5.20)$$

$$L = \left(r\sin\theta\right)p = \left(r\perp\right)p \qquad (5.21)$$

where, $p\perp$ is the component of linear momentum p perpendicular to r, and $r\perp$ is the component of position r perpendicular to p.

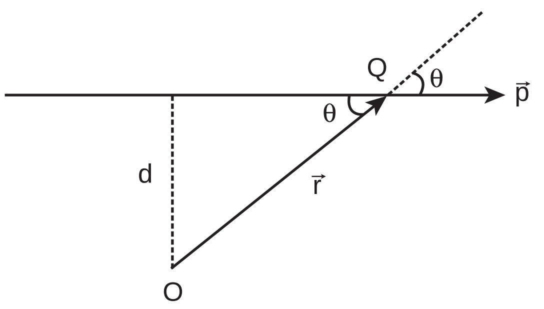
The angular momentum is zero $(L = 0)$, if the linear momentum is zero (p = 0) or if the particle is at the origin $(\vec{r} = 0)$ or if $\vec{r}$ and $\vec{p}$ are parallel or antiparallel to each other (θ = 0^0 or 180^0).

There is a misconception that the angular momentum is a quantity that is associated only with rotational motion. It is not true. The angular momentum is also associated with bodies in the linear motion. Let us understand the same with the following example.

EXAMPLE 5.11

A particle of mass (m) is moving with constant velocity (v). Show that its angular momentum about any point remains constant throughout the motion.

Solution



Let the particle of mass m move with constant velocity $\vec{v}$. As it is moving with constant velocity, its path is a straight line. Its momentum $(\vec{p} = m\vec{v})$ is also directed along the same path. Let us fix an origin (O) at a perpendicular distance (d) from the path. At a particular instant, we can connect the particle which is at positon Q with a position vector $\left(\vec{r} = \overrightarrow{OQ}\right)$.

Take, the angle between the $\vec{r}$ and $\vec{p}$ as θ. The magnitude of angular momentum of that particle at that instant is,

$$L = OQ\,p\sin\theta = OQ\,mv\sin\theta = mv\left(OQ\sin\theta\right)$$

The term $\left(OQ\sin\theta\right)$ is the perpendicular distance (d) between the origin and line along which the mass is moving. Hence, the angular momentum of the particle about the origin is,

$$L = mvd$$

The above expression for angular momentum L, does not have the angle θ. As the momentum (p = mv) and the perpendicular distance (d) are constants, the angular momentum of the particle is also constant. Hence, the angular momentum is associated with bodies with linear motion also. If the straight path of the particle passes through the origin, then the angular momentum is zero, which is also a constant.

5.2.5 Angular Momentum and Angular Velocity

Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about the fixed axis as shown in Figure 5.12.

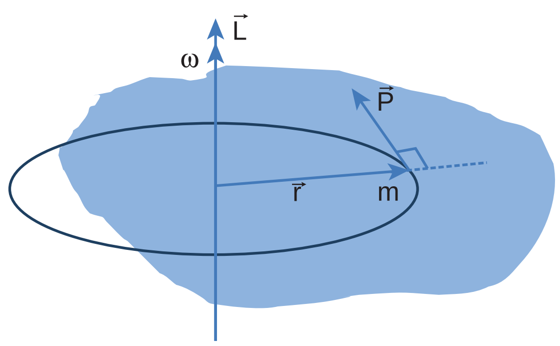


Figure 5.12 Angular momentum and angular velocity

The point mass m is at a distance r from the axis of rotation. Its linear momentum at any instant is tangential to the circular path. Then the angular momentum $\vec{L}$ is perpendicular to $\vec{r}$ and $\vec{p}$. Hence, it is directed along the axis of rotation. The angle θ between $\vec{r}$ and $\vec{p}$ in this case is 90°. The magnitude of the angular momentum L could be written as,

$$L = r\,mv \sin 90^\circ = r\,mv$$

where, v is the linear velocity. The relation between linear velocity v and angular velocity ω in a circular motion is, $v = r\,\omega$. Hence,

$$L = rmr\omega$$

$$L = \left(mr^2\right)\omega \qquad (5.22)$$

The directions of L and ω are along the axis of rotation. The above expression can be written in the vector notation as,

$$\vec{L} = \left(mr^2\right)\vec{\omega} \qquad (5.23)$$

As discussed earlier, the term mr^2 in equations 5.22 and 5.23 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $\left(I = \sum m_i r_i^{\,2}\right)$. Hence, the angular momentum of the rigid body can be written as,

$$\vec{L} = \left(\sum m_i r_i^{\,2}\right)\vec{\omega} \qquad (5.24)$$

$$\vec{L} = I\vec{\omega} \qquad (5.25)$$

The study about moment of inertia (I) is reserved for Section 5.4.

5.2.6 Torque and Angular Momentum

We have the expression for magnitude of angular momentum of a rigid body as, $L = I\omega$. The expression for magnitude of torque on a rigid body is, $\tau = I\alpha$

We can further write the expression for torque as,

$$\tau = I\frac{d\omega}{dt} \quad \because \left(\alpha = \frac{d\omega}{dt}\right) \qquad (5.26)$$

Where, ω is angular velocity and α is angular acceleration. We can also write equation 5.26 as,

$$\tau = \frac{d(I\omega)}{dt}$$

$$\tau = \frac{dL}{dt} \qquad (5.27)$$

The above expression says that an external torque on a rigid body fixed to an axis produces rate of change of angular momentum in the body about that axis. This is the Newton's second law in rotational motion as it is in the form of $F = \frac{dp}{dt}$ which holds good for translational motion.

Conservation of angular momentum:
From the above expression we could conclude that in the absence of external torque, the angular momentum of the rigid body or system of particles is conserved.

$$\text{If } \tau = 0 \text{ then, } \frac{dL}{dt} = 0; \; L = \text{constant}$$

The above expression is known as law of conservation of angular momentum. We will learn about this law further in section 5.5.

5.3 EQUILIBRIUM OF RIGID BODIES

When a body is at rest without any motion on a table, we say that there is no force acting on the body. Actually it is wrong because, there is gravitational force acting on the body downward and also the normal force exerted by table on the body upward. These two forces cancel each other and thus there is no net force acting on the body. There is a lot of difference between the terms "no force" and "no net force" acting on a body. The same argument holds good for rotational conditions in terms of torque or moment of force.

A rigid body is said to be in mechanical equilibrium when both its linear momentum and angular momentum remain constant.

When the linear momentum remains constant, the net force acting on the body is zero.

$$\vec{F}_{net} = 0 \qquad (5.28)$$

In this condition, the body is said to be in translational equilibrium. This implies that the vector sum of different forces $\vec{F}_1, \vec{F}_2, \vec{F}_3 \ldots$ acting in different directions on the body is zero.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n = 0 \qquad (5.29)$$

If the forces $\vec{F}_1, \vec{F}_2, \vec{F}_3 \ldots$ act in different directions on the body, we can resolve them into horizontal and vertical components and then take the resultant in the respective directions. In this case there will be horizontal as well as vertical equilibria possible.

Similarly, when the angular momentum remains constant, the net torque acting on the body is zero.

$$\vec{\tau}_{net} = 0 \qquad (5.30)$$

Under this condition, the body is said to be in rotational equilibrium. The vector sum of different torques $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3 \ldots$ producing different senses of rotation on the body is zero.

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots + \vec{\tau}_n = 0 \qquad (5.31)$$

Thus, we can also conclude that a rigid body is in mechanical equilibrium when the net force and net torque acts on the body is zero.

$$\vec{F}_{net} = 0 \text{ and } \vec{\tau}_{net} = 0 \qquad (5.32)$$

As the forces and torques are vector quantities, the directions are to be taken with proper sign conventions.

5.3.1 Types of Equilibrium

Based on the above discussions, we come to a conclusion that different types of equilibrium are possible based on the different conditions. They are consolidated in Table 5.2.

Table 5.2 Different types of Equilibrium and their Conditions.

Type of equilibrium	Conditions
Translational equilibrium	▪ Linear momentum is constant. ▪ Net force is zero.
Rotational equilibrium	▪ Angular momentum is constant. ▪ Net torque is zero.
Static equilibrium	▪ Linear momentum and angular momentum are zero. ▪ Net force and net torque are zero.
Dynamic equilibrium	▪ Linear momentum and angular momentum are constant. ▪ Net force and net torque are zero.
Stable equilibrium	▪ Linear momentum and angular momentum are zero. ▪ The body tries to come back to equilibrium if slightly disturbed and released. ▪ The centre of mass of the body shifts slightly higher if disturbed from equilibrium. ▪ Potential energy of the body is minimum and it increases if disturbed.
Unstable equilibrium	▪ Linear momentum and angular momentum are zero. ▪ The body cannot come back to equilibrium if slightly disturbed and released. ▪ The centre of mass of the body shifts slightly lower if disturbed from equilibrium. ▪ Potential energy of the body is not minimum and it decreases if disturbed.
Neutral equilibrium	▪ Linear momentum and angular momentum are zero. ▪ The body remains at the same equilibrium if slightly disturbed and released. ▪ The centre of mass of the body does not shift higher or lower if disturbed from equilibrium. ▪ Potential energy remains same even if disturbed.

EXAMPLE 5.12

Arun and Babu carry a wooden log of mass 28 kg and length 10 m which has almost uniform thickness. They hold it at 1 m and 2 m from the ends respectively. Who will bear more weight of the log? [g = 10 ms^{-2}]

Solution

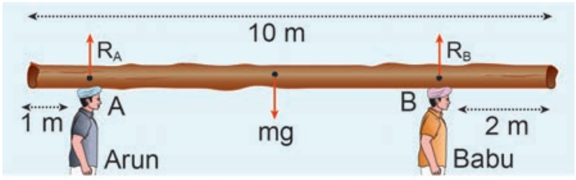
Let us consider the log is in mechanical equilibrium. Hence, the net force and net torque on the log must be zero. The gravitational force acts at the centre of mass of the log downwards. It is cancelled by the normal reaction forces R_A and R_B applied upwards by Arun and Babu at points A and B respectively. These reaction forces are the weights borne by them.

The total weight, W = mg = 28 × 10 = 280 N, has to be borne by them together. The reaction forces are the weights borne by each of them separately. Let us show all the forces acting on the log by drawing a free body diagram of the log.

For translational equilibrium:

The net force acting on the log must be zero.

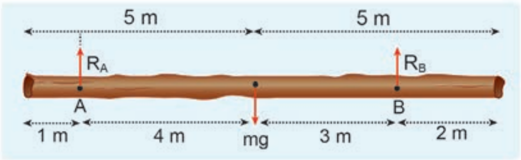
$$R_A + (-mg) + R_B = 0$$



Here, the forces R_A an R_B are taken positive as they act upward. The gravitational force acting downward is taken negative.

$$R_A + R_B = mg$$

For rotational equilibrium:



The net torque acting on the log must be zero. For ease of calculation, we can take the torque caused by all the forces about the point A on the log. The forces are perpendicular to the distances. Hence,

$$(0R_A) + (-4mg) + (7R_B) = 0.$$

Here, the reaction force R_A cannot produce any torque as the reaction forces pass through the point of reference A. The torque of force mg produces a clockwise turn about the point A which is taken negative and torque of force R_B causes anticlockwise turn about A which is taken positive.

$$7R_B = 4mg$$

$$R_B = \frac{4}{7} mg$$

$$R_B = \frac{4}{7} \times 28 \times 10 = 160 \text{ N}$$

By substituting for R_B we get,

$$R_A = mg - R_B$$

$$R_A = 28 \times 10 - 160 = 280 - 160 = 120 \text{ N}$$

As R_B is greater than R_A, it is concluded that Babu bears more weight than Arun. The one closer to centre of mass of the log bears more weight.

5.3.2 Couple

Consider a thin uniform rod AB. Its centre of mass is at its midpoint C. Let two forces which are equal in magnitude and opposite in direction be applied at the two ends A and B of the rod perpendicular to it. The two forces are separated by a distance of 2r as shown in Figure 5.13.

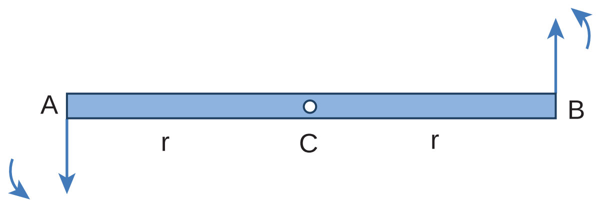


Figure 5.13 Couple

As the two equal forces are opposite in direction, they cancel each other and the net force acting on the rod is zero. Now the rod is in translational equilibrium. But, the rod is not in rotational equilibrium. Let us see how it is not in rotational equilibrium. The moment of the force applied at the end A taken with respect to the centre point C, produces an anticlockwise rotation. Similarly, the moment of the force applied at the end B also produces an anticlockwise rotation. The moments of both the forces cause the same sense of rotation in the rod. Thus, the rod undergoes a rotational motion or turning even though the rod is in translational equilibrium.

A pair of forces which are equal in magnitude but opposite in direction and separated by a perpendicular distance so that their lines of action do not coincide that causes a turning effect is called a couple. We come across couple in many of our daily activities as shown in Figure 5.14.

> **Note** There are cases in which the two forces may not cancel each other. If the two forces are not equal or the direction of the forces is not exactly opposite, then the body will have both translational as well as rotational motion.

5.3.3 Principle of Moments

Consider a light rod of negligible mass which is pivoted at a point along its length. Let two parallel forces F_1 and F_2 act at the two ends at distances d_1 and d_2 from the point of pivot and the normal reaction force N at

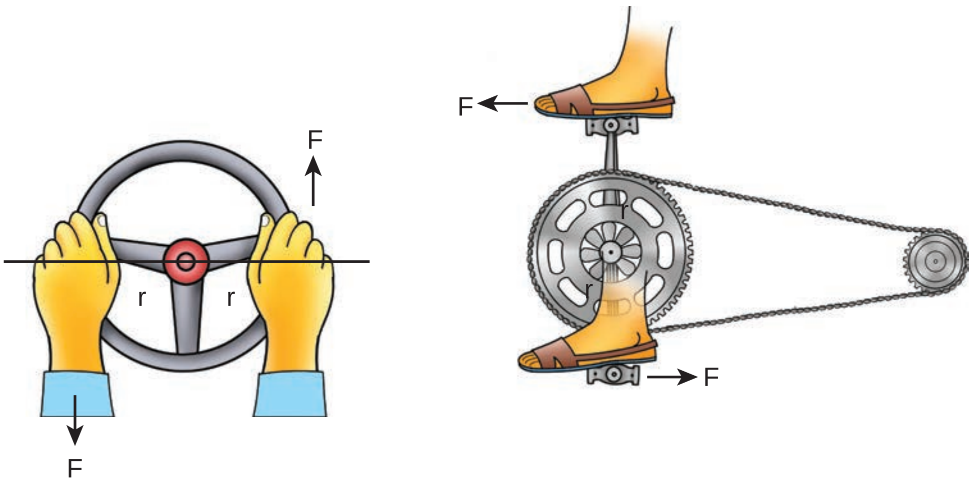


Figure 5.14 Turning effect of Couple

the point of pivot as shown in Figure 5.15. If the rod has to remain stationary in horizontal position, it should be in translational and rotational equilibrium. Then, both the net force and net torque must be zero.

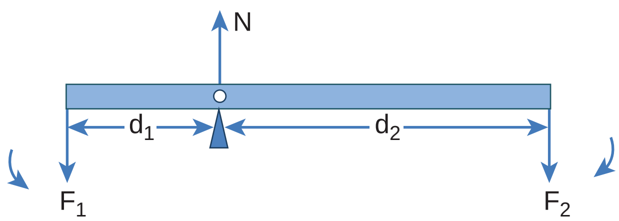


Figure 5.15 Principle of Moments

For translational equilibrium,
net force has to be zero, $-F_1 + N - F_2 = 0$

$$N = F_1 + F_2$$

For rotational equilibrium,
net torque has to be zero, $d_1F_1 - d_2F_2 = 0$

$$d_1F_1 = d_2F_2 \qquad (5.33)$$

The above equation represents the *principle of moments*. This forms the principle for beam balance used for weighing goods with the condition $d_1 = d_2$; $F_1 = F_2$. We can rewrite the equation 5.33 as,

$$\frac{F_1}{F_2} = \frac{d_2}{d_1} \qquad (5.34)$$

If F_1 is the load and F_2 is our effort, we get advantage when, $d_1 < d_2$. This implies that $F_1 > F_2$. Hence, we could lift a large load with small effort. The ratio $\left(\frac{d_2}{d_1}\right)$ is called mechanical advantage of the simple lever. The pivoted point is called fulcrum.

$$\text{Mechanical Advantage}(MA) = \frac{d_2}{d_1} \qquad (5.35)$$

There are many simple machines that work on the above mentioned principle.

5.3.4 Centre of Gravity

Each rigid body is made up of several point masses. Such point masses experience gravitational force towards the centre of Earth. As the size of Earth is very large compared to any practical rigid body we come across in daily life, these forces appear to be acting parallelly downwards as shown in Figure 5.16.

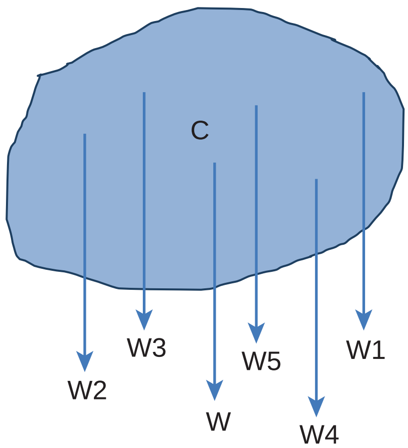


Figure 5.16. Centre of gravity

The resultant of these parallel forces always acts through a point. This point is called centre of gravity of the body (with respect to Earth). The *centre of gravity of a body is the point at which the entire weight of the body acts irrespective of the position and orientation of the body*. The centre of gravity and centre of mass of a rigid body coincide when the gravitational field is uniform across the body. The concept of gravitational field is dealt in Unit 6.

We can also determine the centre of gravity of a uniform lamina of even an irregular shape by pivoting it at various points by trial and error. The lamina remains horizontal when pivoted at the point where the net gravitational force acts, which is the centre of gravity as shown in Figure 5.17. When a body is supported at the centre of gravity, the sum of the torques acting on all the point masses of the rigid body becomes zero. Moreover the weight is compensated by the normal reaction force exerted by the pivot. The body is in static equilibrium and hence it remains horizontal.

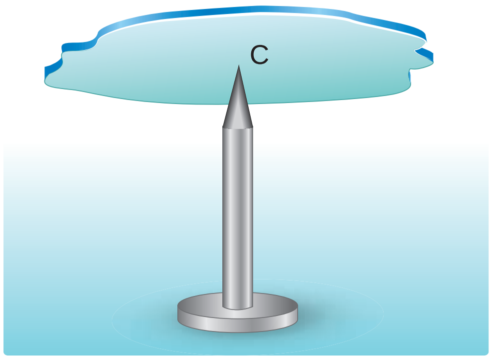


Figure 5.17. Determination of centre of gravity of plane lamina by pivoting

There is also another way to determine the centre of gravity of an irregular lamina. If we suspend the lamina from different points like P, Q, R as shown in Figure 5.18, the vertical lines PP', QQ', RR' all pass through the centre of gravity. Here, reaction force acting at the point of suspension and the gravitational force acting at the centre of gravity cancel each other and the torques caused by them also cancel each other.

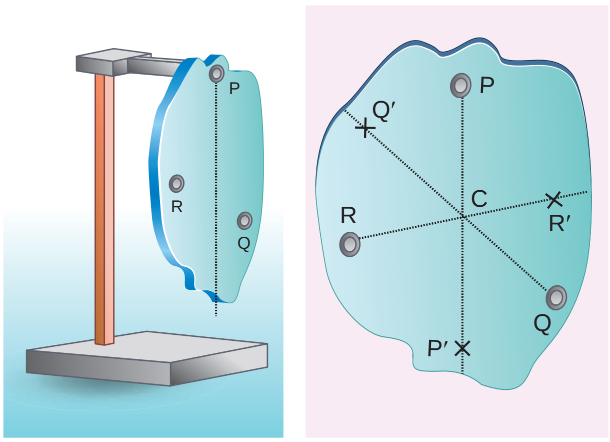


Figure 5.18 Determination of centre of gravity of plane lamina by suspending

5.3.5 Bending of Cyclist in Curves

Let us consider a cyclist negotiating a circular level road (not banked) of radius r with a speed v. The cycle and the cyclist are considered as one system with mass m. The centre gravity of the system is C and it goes in a circle of radius r with centre at O. Let us choose the line OC as X-axis and the vertical line through O as Z-axis as shown in Figure 5.19.

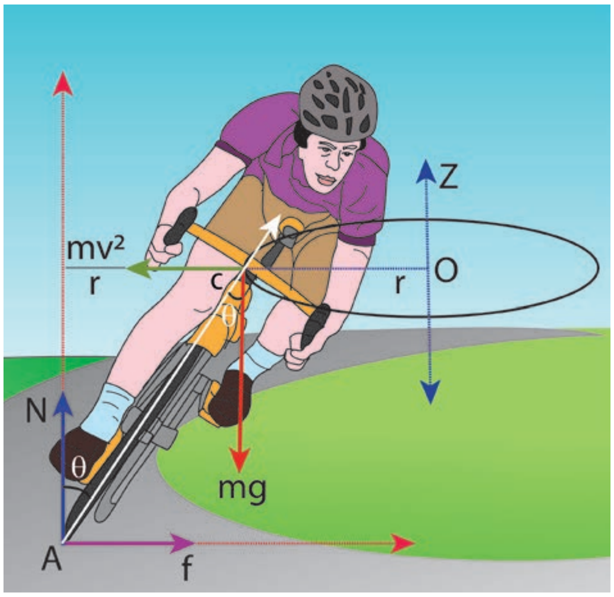


Figure 5.19 Bending of cyclist

The system as a frame is rotating about Z-axis. The system is at rest in this rotating frame. To solve problems in rotating frame of reference, we have to apply a centrifugal force (pseudo force) on the system which will be $\frac{mv^2}{r}$. This force will act through the centre of gravity. The forces acting on the system are, (i) gravitational force (mg), (ii) normal force (N), (iii) frictional force (f) and (iv) centrifugal force $\left(\frac{mv^2}{r}\right)$. As the system is in equilibrium in the rotational frame of reference, the net external force and net external torque must be zero. Let us consider all torques about the point A in Figure 5.20.

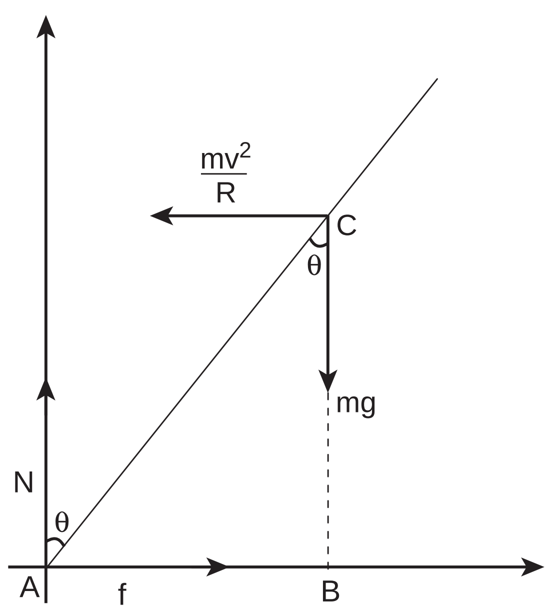


Figure 5.20 Force diagrams for the cyclist in turns

For rotational equilibrium,

$$\vec{\tau}_{net} = 0$$

The torque due to the gravitational force about point A is $(mg\,AB)$ which causes a clockwise turn that is taken as negative. The torque due to the centrifugal force is $\left(\frac{mv^2}{r}BC\right)$ which causes an anticlockwise turn that is taken as positive.

$$-mg\,AB + \frac{mv^2}{r}BC = 0$$

$$mg\,AB = \frac{mv^2}{r}BC$$

From Δ ABC,

$$AB = AC\sin\theta \text{ and } BC = AC\cos\theta$$

$$mg\,AC\sin\theta = \frac{mv^2}{r}AC\cos\theta$$

$$\tan\theta = \frac{v^2}{rg}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \quad (5.36)$$

While negotiating a circular level road of radius r at velocity v, a cyclist has to bend by an angle θ from vertical given by the above expression to stay in equilibrium (i.e. to avoid a fall).

EXAMPLE 5.13

A cyclist while negotiating a circular path with speed 20 m s^{-1} is found to bend an angle by 30° with vertical. What is the radius of the circular path? (given, $g = 10$ m s^{-2})

Solution

Speed of the cyclist, $v = 20\ m\ s^{-1}$
Angle of bending with vertical, $\theta = 30^{o}$

Equation for angle of bending, $\tan\theta = \frac{v^2}{rg}$

Rewriting the above equation for radius

$r = \frac{v^2}{\tan\theta\, g}$

Substituting,

$$r = \frac{(20)^2}{\left(\tan 30^{\circ}\right)\times 10} = \frac{20\times 20}{\left(\tan 30^{\circ}\right)\times 10}$$

$$= \frac{400}{\left(\frac{1}{\sqrt{3}}\right)\times 10}$$

$$r = \left(\sqrt{3}\right)\times 40 = 1.732\times 40$$

$$r = 69.28\,m$$

5.4 MOMENT OF INERTIA

In the expressions for torque and angular momentum for rigid bodies (which are considered as bulk objects), we have come across a term $\sum m_i r_i^2$. This quantity is called moment of inertia (I) of the bulk object. For point mass m_i at a distance r_i from the fixed axis, the moment of inertia is given as, $m_i r_i^2$.

Moment of inertia for point mass,

$$I = m_i r_i^2 \qquad (5.37)$$

Moment of inertia for bulk object,

$$I = \sum m_i r_i^2 \qquad (5.38)$$

In translational motion, mass is a measure of inertia; in the same way, for rotational motion, moment of inertia is a measure of rotational inertia. The unit of moment of inertia is, kg m². Its dimension is ML^2. In general, mass is an invariable quantity of matter (except for motion comparable to that of light). But, the moment of inertia of a body is not an invariable quantity. It depends not only on the mass of the body, but also on the way the mass is distributed around the axis of rotation.

To find the moment of inertia of a uniformly distributed mass; we have to consider an infinitesimally small mass (dm) as a point mass and take its position (r) with respect to an axis. The moment of inertia of this point mass can now be written as,

$$dI = (dm)\,r^2 \qquad (5.39)$$

We get the moment of inertia of the entire bulk object by integrating the above expression.

$$I = \int dI = \int (dm)\,r^2$$

$$I = \int r^2\,dm \qquad (5.40)$$

We can use the above expression for determining the moment of inertia of some of the common bulk objects of interest like rod, ring, disc, sphere etc.

5.4.1 Moment of Inertia of a Uniform Rod

Let us consider a uniform rod of mass (M) and length (ℓ) as shown in Figure 5.21. Let us find an expression for moment of inertia of this rod about an axis that passes through the centre of mass and perpendicular to

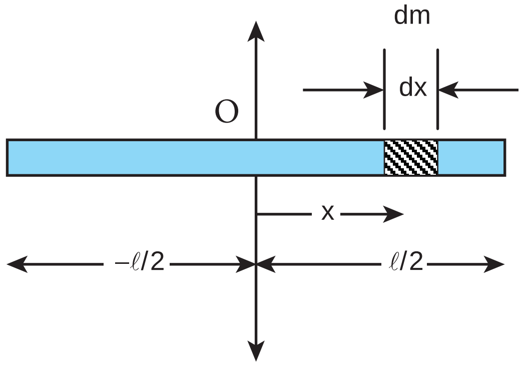


Figure 5.21 Moment of inertia of uniform rod

the rod. First an origin is to be fixed for the coordinate system so that it coincides with the centre of mass, which is also the geometric centre of the rod. The rod is now along the x axis. We take an infinitesimally small mass (dm) at a distance (x) from the origin. The moment of inertia (dI) of this mass (dm) about the axis is,

$$dI = (dm)x^2$$

As the mass is uniformly distributed, the mass per unit length (λ) of the rod is, $\lambda = \frac{M}{\ell}$

The (dm) mass of the infinitesimally small length as, $dm = \lambda\, dx = \frac{M}{\ell}dx$

The moment of inertia (I) of the entire rod can be found by integrating dI,

$$I = \int dI = \int (dm)x^2 = \int \left(\frac{M}{\ell}dx\right)x^2$$

$$I = \frac{M}{\ell}\int x^2 dx$$

As the mass is distributed on either side of the origin, the limits for integration are taken from $-\ell/2$ to $\ell/2$.

$$I = \frac{M}{\ell}\int_{-\ell/2}^{\ell/2} x^2 dx = \frac{M}{\ell}\left[\frac{x^3}{3}\right]_{-\ell/2}^{\ell/2}$$

$$I = \frac{M}{\ell}\left[\frac{\ell^3}{24} - \left(-\frac{\ell^3}{24}\right)\right] = \frac{M}{\ell}\left[\frac{\ell^3}{24} + \frac{\ell^3}{24}\right]$$

$$I = \frac{M}{\ell}\left[2\left(\frac{\ell^3}{24}\right)\right]$$

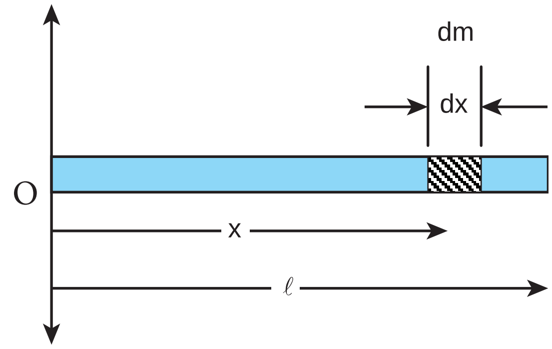
$$I = \frac{1}{12}M\ell^2 \qquad (5.41)$$

EXAMPLE 5.14

Find the moment of inertia of a uniform rod about an axis which is perpendicular to the rod and touches any one end of the rod.

Solution

The concepts to form the integrand to find the moment of inertia could be borrowed from the earlier derivation. Now, the origin is fixed to the left end of the rod and the limits are to be taken from 0 to ℓ.



$$I = \frac{M}{\ell}\int_0^{\ell} x^2 dx = \frac{M}{\ell}\left[\frac{x^3}{3}\right]_0^{\ell} = \frac{M}{\ell}\left[\frac{\ell^3}{3}\right]$$

$$I = \frac{1}{3}M\ell^2$$

Note: The moment of inertia of the same uniform rod is different about different axes of reference. The reference axes could be even outside the object. We have two useful theorems to calculate the moments of inertia about different axes. We shall see these theorems in Section 5.4.5.

5.4.2 Moment of Inertia of a Uniform Ring

Let us consider a uniform ring of mass M and radius R. To find the moment of inertia of the ring about an axis passing through its centre and perpendicular to the plane, let us take an infinitesimally small mass (dm) of length (dx) of the ring. This (dm) is located at a distance R, which is the radius of the ring from the axis as shown in Figure 5.22.

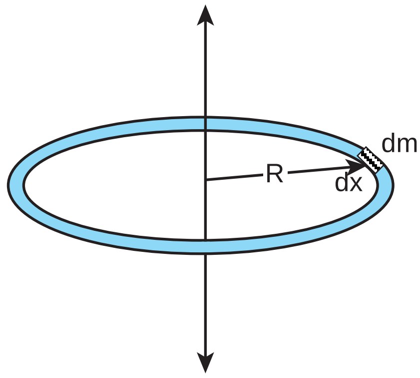


Figure 5.22 Moment of inertia of a uniform ring

The moment of inertia (dI) of this small mass (dm) is,

$$dI = (dm)R^2$$

The length of the ring is its circumference $(2\pi R)$. As the mass is uniformly distributed, the mass per unit length (λ) is,

$$\lambda = \frac{\text{mass}}{\text{length}} = \frac{M}{2\pi R}$$

The mass (dm) of the infinitesimally small length is, $dm = \lambda\, dx = \frac{M}{2\pi R} dx$

Now, the moment of inertia (I) of the entire ring is,

$$I = \int dI = \int (dm)R^2 = \int \left(\frac{M}{2\pi R} dx\right) R^2$$

$$I = \frac{MR}{2\pi} \int dx$$

To cover the entire length of the ring, the limits of integration are taken from 0 to $2\pi R$.

$$I = \frac{MR}{2\pi} \int_0^{2\pi R} dx$$

$$I = \frac{MR}{2\pi} [x]_0^{2\pi R} = \frac{MR}{2\pi} [2\pi R - 0]$$

$$I = MR^2 \qquad (5.42)$$

5.4.3 Moment of Inertia of a Uniform Disc

Consider a disc of mass M and radius R. This disc is made up of many infinitesimally small rings as shown in Figure 5.23. Consider one such ring of mass (dm) and thickness (dr) and radius (r). The moment of inertia (dI) of this small ring is,

$$dI = (dm)r^2$$

As the mass is uniformly distributed, the mass per unit area (σ) is, $\sigma = \frac{\text{mass}}{\text{area}} = \frac{M}{\pi R^2}$

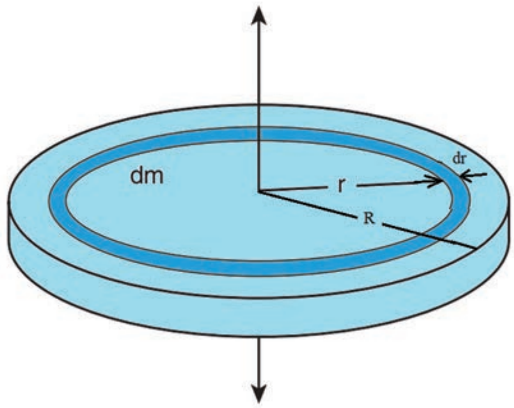


Figure 5.23 Moment of inertia of a uniform disc

The mass of the infinitesimally small ring is,

$$dm = \sigma 2\pi r dr = \frac{M}{\pi R^2} 2\pi r dr$$

where, the term $(2\pi r\, dr)$ is the area of this elemental ring ($2\pi r$ is the length and dr is the thickness). $dm = \frac{2M}{R^2} r dr$

$$dI = \frac{2M}{R^2} r^3 dr$$

The moment of inertia (I) of the entire disc is,

$$I = \int dI$$

$$I = \int_0^R \frac{2M}{R^2} r^3 dr = \frac{2M}{R^2} \int_0^R r^3 dr$$

$$I = \frac{2M}{R^2} \left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2} \left[\frac{R^4}{4} - 0\right]$$

$$I = \frac{1}{2} MR^2 \qquad (5.43)$$

5.4.4 Radius of Gyration

For bulk objects of regular shape with uniform mass distribution, the expression for moment of inertia about an axis involves their total mass and geometrical features like radius, length, breadth, which take care of the shape and the size of the objects. But, we need an expression for the moment of inertia which could take care of not only the mass, shape and size of objects, but also its orientation to the axis of rotation. Such an expression should be general so that it is applicable even for objects of irregular shape and non-uniform distribution of mass. The general expression for moment of inertia is given as,

$$I = MK^2 \qquad (5.44)$$

where, M is the total mass of the object and K is called the radius of gyration.

The radius of gyration of an object is the perpendicular distance from the axis of rotation to an equivalent point mass, which would have the same mass as well as the same moment of inertia of the object.

As the radius of gyration is distance, its unit is m. Its dimension is [L].

A rotating rigid body with respect to any axis, is considered to be made up of point masses $m_1, m_2, m_3, \ldots m_n$ at perpendicular distances (or positions) $r_1, r_2, r_3 \ldots r_n$ respectively as shown in Figure 5.24.

The moment of inertia of that object can be written as,

$$I = \sum_{i=1}^{n} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2$$

If we take all the n number of individual masses to be equal,

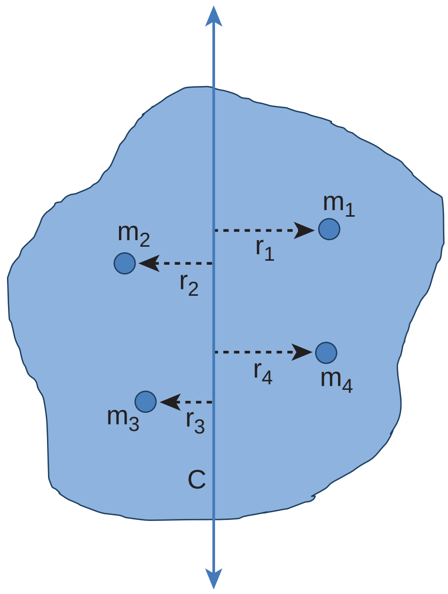


Figure 5.24 Radius of gyration

$$m = m_1 = m_2 = m_3 = \ldots = m_n$$

then,

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$$
$$= m\left(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2\right)$$
$$= nm\left(\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}\right)$$
$$I = MK^2$$

where, nm is the total mass M of the body and K is the radius of gyration.

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}} \qquad (5.45)$$

The expression for radius of gyration indicates that it is the root mean square (rms) distance of the particles of the body from the axis of rotation.

In fact, the moment of inertia of any object could be expressed in the form, $I = MK^2$.

For example, let us take the moment of inertia of a uniform rod of mass M and length ℓ. Its moment of inertia with respect to a perpendicular axis passing through the centre of mass is, $I = \frac{1}{12}M\ell^2$

In terms of radius of gyration, $I = MK^2$

$$\text{Hence, } MK^2 = \frac{1}{12}M\ell^2$$
$$K^2 = \frac{1}{12}\ell^2$$
$$K = \frac{1}{\sqrt{12}}\ell \text{ or } K = \frac{1}{2\sqrt{3}}\ell \text{ or } K = (0.289)\ell$$

EXAMPLE 5.15

Find the radius of gyration of a disc of mass M and radius R rotating about an axis passing through the centre of mass and perpendicular to the plane of the disc.

Solution

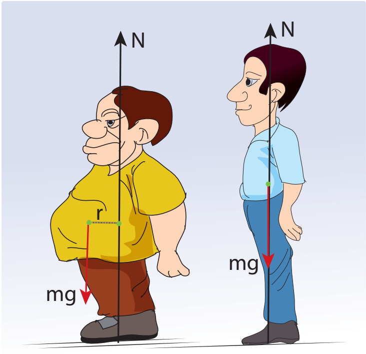
The moment of inertia of a disc about an axis passing through the centre of mass and perpendicular to the disc is, $I = \frac{1}{2}MR^2$

In terms of radius of gyration, $I = MK^2$

$$\text{Hence, } MK^2 = \frac{1}{2}MR^2; \quad K^2 = \frac{1}{2}R^2$$
$$K = \frac{1}{\sqrt{2}}R \text{ or } K = \frac{1}{1.414}R \text{ or } K = (0.707)R$$

From the case of a rod and also a disc, we can conclude that the radius of gyration of the rigid body is always a geometrical feature like length, breadth, radius or their combinations with a positive numerical value multiplied to it.

Obesity, torque and Moment of Inertia!



Obesity and associated ailments like back pain, joint pain etc. are due to the shift in centre of mass of the body. Due to this shift in centre of mass, unbalanced torque acting on the body leads to ailments. As the mass is spread away from centre of the body the moment of inertia is more and turning will also be difficult.

5.4.5 Theorems of Moment of Inertia

As the moment of inertia depends on the axis of rotation and also the orientation of the body about that axis, it is different for the same body with different axes of rotation. We have two important theorems to handle the case of shifting the axis of rotation.

(i) Parallel axis theorem:

Parallel axis theorem states that the moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of mass and the product of the mass of the body and the square of the perpendicular distance between the two axes.

If I_C is the moment of inertia of the body of mass M about an axis passing through the centre of mass, then the moment of inertia I about a parallel axis at a distance d from it is given by the relation,

$$I = I_C + Md^2 \qquad (5.46)$$

Let us consider a rigid body as shown in Figure 5.25. Its moment of inertia about an axis AB passing through the centre of mass is I_C. DE is another axis parallel to AB at a perpendicular distance d from AB. The moment of inertia of the body about DE is I. We attempt to get an expression for I in terms of I_C. For this, let us consider a point mass m on the body at position x from its centre of mass.

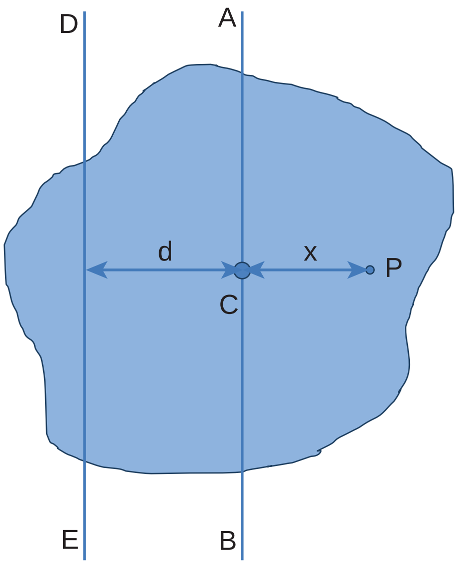


Figure 5.25 Parallel axis theorem

The moment of inertia of the point mass about the axis DE is, $m(x+d)^2$.

The moment of inertia I of the whole body about DE is the summation of the above expression.

$$I = \sum m(x+d)^2$$

This equation could further be written as,

$$I = \sum m\left(x^2 + d^2 + 2xd\right)$$

$$I = \sum\left(mx^2 + md^2 + 2dmx\right)$$

$$I = \sum mx^2 + \sum md^2 + 2d\sum mx$$

Here, $\sum mx^2$ is the moment of inertia of the body about the centre of mass. Hence, $I_C = \sum mx^2$

The term, $\sum mx = 0$ because, x can take positive and negative values with respect to the axis AB. The summation $\left(\sum mx\right)$ will be zero.

Thus, $I = I_C + \sum md^2 = I_C + \left(\sum m\right)d^2$

Here, $\sum m$ is the entire mass M of the object $\left(\sum m = M\right)$

$$I = I_C + Md^2$$

Hence, the parallel axis theorem is proved.

(ii) Perpendicular axis theorem:
This perpendicular axis theorem holds good only for plane laminar objects.

The theorem states that the moment of inertia of a plane laminar body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two perpendicular axes lying in the plane of the body such that all the three axes are mutually perpendicular and have a common point.

Let the X and Y-axes lie in the plane and Z-axis perpendicular to the plane of the laminar object. If the moments of inertia of the body about X and Y-axes are I_X and I_Y respectively and I_Z is the moment of inertia about Z-axis, then the perpendicular axis theorem could be expressed as,

$$I_Z = I_X + I_Y \tag{5.47}$$

To prove this theorem, let us consider a plane laminar object of negligible thickness on which lies the origin (O). The X and Y-axes lie on the plane and Z-axis is perpendicular to it as shown in Figure 5.26. The lamina is considered to be made up of a large number of particles of mass m. Let us choose one such particle at a point P which has coordinates (x, y) at a distance r from O.

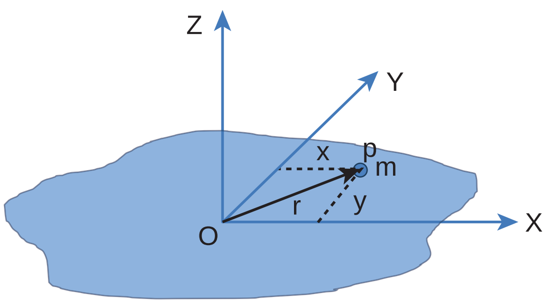


Figure 5.26 Perpendicular axis theorem

The moment of inertia of the particle about Z-axis is, mr^2

The summation of the above expression gives the moment of inertia of the entire lamina about Z-axis as, $I_Z = \sum mr^2$

Here, $r^2 = x^2 + y^2$

Then, $I_Z = \sum m\left(x^2 + y^2\right)$

$$I_Z = \sum mx^2 + \sum my^2$$

In the above expression, the term $\sum mx^2$ is the moment of inertia of the body about the

UNIT-5(XI-Physics_Vol-1).indd 240 06-01-2022 17:47:26

Y-axis and similarly the term $\sum my^2$ is the moment of inertia about X-axis. Thus,

$$I_X = \sum my^2 \quad \text{and} \quad I_Y = \sum mx^2$$

Substituting in the equation for I_z gives,

$$I_Z = I_X + I_Y$$

Thus, the perpendicular axis theorem is proved.

EXAMPLE 5.16

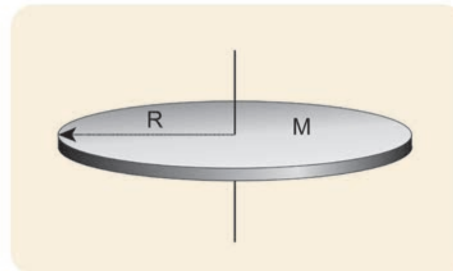
Find the moment of inertia of a disc of mass 3 kg and radius 50 cm about the following axes.

(i) axis passing through the centre and perpendicular to the plane of the disc,
(ii) axis touching the edge and perpendicular to the plane of the disc and
(iii) axis passing through the centre and lying on the plane of the disc.

Solution

The mass, M = 3 kg, radius R = 50 cm = 50×10^{-2} m = 0.5 m

(i) The moment of inertia (I) about an axis passing through the centre and perpendicular to the plane of the disc is,

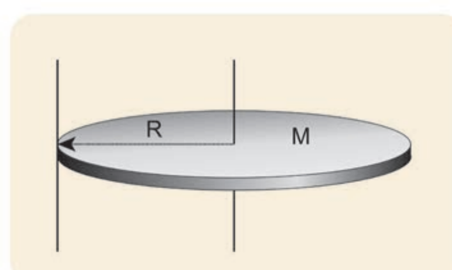


$$I = \frac{1}{2} MR^2$$

$$I = \frac{1}{2} \times 3 \times (0.5)^2 = 0.5 \times 3 \times 0.5 \times 0.5$$

$$I = 0.375 \, \text{kg m}^2$$

(ii) The moment of inertia (I) about an axis touching the edge and perpendicular to the plane of the disc by parallel axis theorem is,



$$I = I_C + Md^2$$

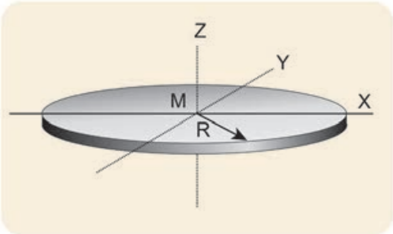
where, $I_C = \frac{1}{2} MR^2$ and $d = R$

$$I = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$$

$$I = \frac{3}{2} \times 3 \times (0.5)^2 = 1.5 \times 3 \times 0.5 \times 0.5$$

$$I = 1.125 \, \text{kg m}^2$$

(iii) The moment of inertia (I) about an axis passing through the centre and lying on the plane of the disc is,



$$I_Z = I_X + I_Y$$

where, $I_X = I_Y = I$ and $I_Z = \frac{1}{2}MR^2$

$$I_Z = 2I;\ I = \frac{1}{2}I_Z$$

$$I = \frac{1}{2} \times \frac{1}{2}MR^2 = \frac{1}{4}MR^2$$

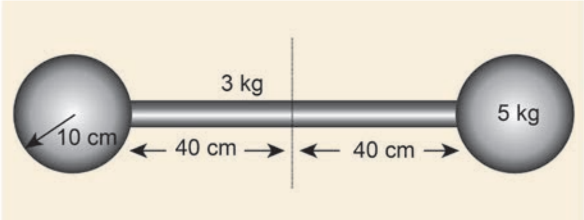
$$I = \frac{1}{4} \times 3 \times (0.5)^2 = 0.25 \times 3 \times 0.5 \times 0.5$$

$$I = 0.1875\,\text{kg m}^2$$

- About which of the above axis it is easier to rotate the disc?
- It is easier to rotate the disc about an axis about which the moment of inertia is the least. Hence, it is case (iii).

EXAMPLE 5.17

Find the moment of inertia about the geometric centre of the given structure made up of one thin rod connecting two similar solid spheres as shown in Figure.



Solution

The structure is made up of three objects; one thin rod and two solid spheres.

The mass of the rod, M = 3 kg and the total length of the rod, $\ell = 80\text{ cm} = 0.8\text{ m}$

The moment of inertia of the rod about its centre of mass is, $I_{rod} = \frac{1}{12}M\ell^2$

$$I_{rod} = \frac{1}{12} \times 3 \times (0.8)^2 = \frac{1}{4} \times 0.64$$

$$I_{rod} = 0.16\,\text{kg m}^2$$

The mass of the sphere, M = 5 kg and the radius of the sphere, R = 10 cm = 0.1 m

The moment of inertia of the sphere about its centre of mass is, $I_C = \frac{2}{5}MR^2$

The moment of inertia of the sphere about geometric centre of the structure is,

$I_{sph} = I_C + Md^2$

Where, d = 40 cm + 10 cm = 50 cm = 0.5 m

$$I_{sph} = \frac{2}{5}MR^2 + Md^2$$

$$I_{sph} = \frac{2}{5} \times 5 \times (0.1)^2 + 5 \times (0.5)^2$$

$$I_{sph} = (2 \times 0.01) + (5 \times 0.25) = 0.02 + 1.25$$

$$I_{sph} = 1.27\,\text{kg m}^2$$

As there are one rod and two similar solid spheres we can write the total moment of inertia (I) of the given geometric structure as, $I = I_{rod} + (2 \times I_{sph})$

$$I = (0.16) + (2 \times 1.27) = 0.16 + 2.54$$

$$I = 2.7\,\text{kg m}^2$$

5.4.6 Moment of Inertia of Different Rigid Bodies

The moment of inertia of different objects about different axes is given in the Table 5.3.

5.5 ROTATIONAL DYNAMICS

The relations among torque, angular acceleration, angular momentum, angular velocity and moment of inertia were seen in Section 5.2. In continuation to that, in this section, we will learn the relations among the other dynamical quantities like work, kinetic energy in rotational motion of rigid bodies. Finally a comparison between the translational and rotational quantities is made with a tabulation.

5.5.1 Effect of Torque on Rigid Bodies

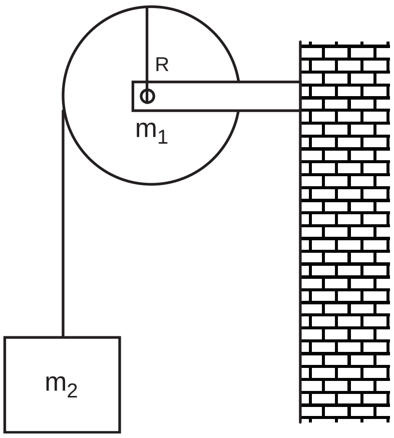
A rigid body which has non zero external torque (τ) about the axis of rotation would have an angular acceleration (α) about that axis. The scalar relation between the torque and angular acceleration is,

$$\tau = I\alpha \tag{5.48}$$

where, I is the moment of inertia of the rigid body. The torque in rotational motion is equivalent to the force in linear motion.

EXAMPLE 5.18

A disc of mass 500 g and radius 10 cm can freely rotate about a fixed axis as shown in figure. light and inextensible string is wound several turns around it and 100 g body is suspended at its free end. Find the acceleration of this mass. [Given: The string makes the disc to rotate and does not slip over it. $g = 10 \text{ m s}^{-2}$.]



Solution

Let the mass of the disc be m_1 and its radius R. The mass of the suspended body is m_2.

$$m_1 = 500 \text{ g} = 500\times10^{-3} \text{ kg} = 0.5 \text{ kg}$$
$$m_2 = 100 \text{ g} = 100\times10^{-3} \text{ kg} = 0.1 \text{ kg}$$
$$R = 10 \text{ cm} = 10\times10^{-2} \text{ m} = 0.1 \text{ m}$$

As the light inextensible string is wound around the disc several times it makes the disc rotate without slipping over it. The translational acceleration of m_2 and tangential acceleration of m_1 will be the same. Let us draw the free body diagram (*FBD*) of m_1 and m_2 separately.

FBD of the disc:

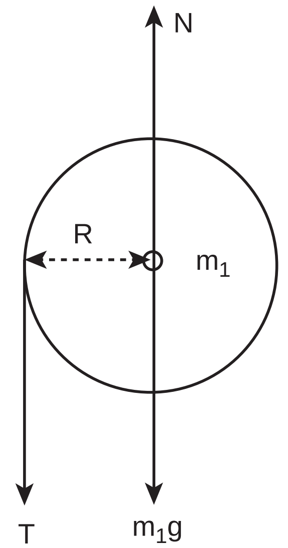
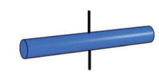
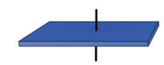
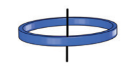
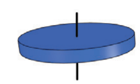
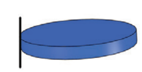
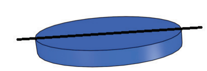
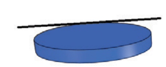
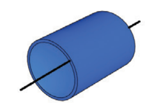
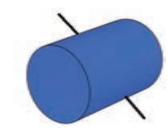
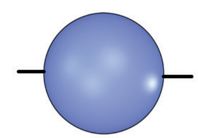
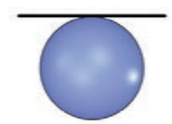
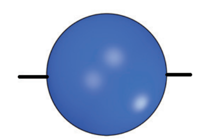
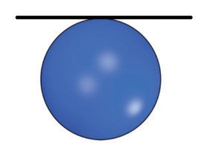


Table 5.3 Moment of Inertia of Different Rigid Bodies

No.	Object	About an axis	Diagram	Moment of Inertia (I) kg m^2	Radius of Gyration (K)	Ratio $\left(\frac{K^2}{R^2}\right)$
1.	**Thin Uniform Rod** Mass = M Length = ℓ	Passing through the centre and perpendicular to the length		$\frac{1}{12}M\ell^2$	$\frac{\ell}{\sqrt{12}}$	--
		Touching one end and perpendicular to the length		$\frac{1}{3}M\ell^2$	$\frac{\ell}{\sqrt{3}}$	
2.	**Thin Uniform Rectangular Sheet** Mass = M; Length = ℓ; Breadth = b	Passing through the centre and perpendicular to the plane of the sheet		$\frac{1}{12}M(\ell^2+b^2)$	$\sqrt{\frac{(\ell^2+b^2)}{12}}$	--
3.	Thin Uniform Ring Mass = M Radius = R	Passing through the centre and perpendicular to the plane		MR^2	R	1
		Touching the edge perpendicular to the plane (perpendicular tangent)		$2MR^2$	$\left(\sqrt{2}\right)R$	2
		Passing through the centre lying on the plane (along diameter)		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Touching the edge parallel to the plane (parallel tangent)		$\frac{3}{2}MR^2$	$\left(\sqrt{\frac{3}{2}}\right)R$	$\frac{3}{2}$
4.	**Thin Uniform Disc** Mass = M Radius = R	Passing through the centre and perpendicular to the plane		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Touching the edge perpendicular to the plane (perpendicular tangent to the plane)		$\frac{3}{2}MR^2$	$\left(\sqrt{\frac{3}{2}}\right)R$	$\frac{3}{2}$
		Passing through the centre lying on the plane (along diameter)		$\frac{1}{4}MR^2$	$\left(\frac{1}{2}\right)R$	$\frac{1}{4}$
		Touching the edge parallel to the plane (parallel tangent to the plane)		$\frac{5}{4}MR^2$	$\left(\sqrt{\frac{5}{4}}\right)R$	$\frac{5}{4}$

5.	**Thin Uniform Hollow Cylinder** Mass = M Length = ℓ; Radius = R	Passing through the centre and along the axis of the cylinder		MR^2	R	1
		Passing perpendicular to the length and passing through the centre		$M\left(\frac{R^2}{2}+\frac{\ell^2}{12}\right)$	$\sqrt{\frac{R^2}{2}+\frac{\ell^2}{12}}$	- -
6.	Uniform Solid Cylinder Mass = M Length = ℓ; Radius = R	Passing through the centre and along the axis of the cylinder		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Passing perpendicular to the length and passing through the centre		$M\left(\frac{R^2}{4}+\frac{\ell^2}{12}\right)$	$\sqrt{\frac{R^2}{4}+\frac{\ell^2}{12}}$	- -
7.	Thin Hollow Sphere (Thin Spherical Shell) Mass = M Radius = R	Passing through the centre (along diameter)		$\frac{2}{3}MR^2$	$\left(\sqrt{\frac{2}{3}}\right)R$	$\frac{2}{3}$
		Touching the edge (tangent)		$\frac{5}{3}MR^2$	$\left(\sqrt{\frac{5}{3}}\right)R$	$\frac{5}{3}$
8.	Uniform Solid Sphere Mass = M Radius = R	Passing through the centre (along diameter)		$\frac{2}{5}MR^2$	$\left(\sqrt{\frac{2}{5}}\right)R$	$\frac{2}{5}$
		Touching the edge (tangent)		$\frac{7}{5}MR^2$	$\left(\sqrt{\frac{7}{5}}\right)R$	$\frac{7}{5}$

Its gravitational force (m_1g) acts downward and normal force N exerted by the fixed support at the centre acts upward. The tension T acts downward at the edge. The gravitational force (m_1g) and the normal force (N) cancel each other. $m_1g = N$

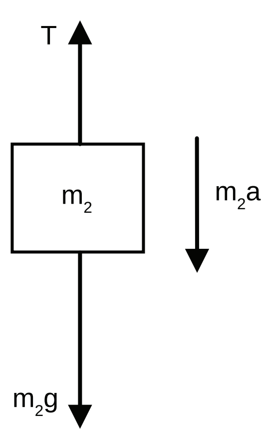
The tension T produces a torque (R T), which produces a rotational motion in the disc with angular acceleration, $\left(\alpha = \frac{a}{R}\right)$. Here, a is the linear acceleration of a point at the edge of the disc. If the moment of inertia of the disc is I and its radius of gyration is K, then

$$R\,T = I\alpha; \quad R\,T = \left(m_1K^2\right)\frac{a}{R}$$

$$T = \left(m_1K^2\right)\frac{a}{R^2}$$

FBD of the body:

Its gravitational force (m_2g) acts downward and the tension T acts upward. As ($T < m_2g$), there is a resultant force (m_2a) acting on it downward.



$$m_2g - T = m_2a$$

Substituting for T from the equation for disc,

$$m_2g - \left(m_1K^2\right)\frac{a}{R^2} = m_2a$$

$$m_2g = \left(m_1K^2\right)\frac{a}{R^2} + m_2a$$

$$m_2g = \left[\left(m_1\frac{K^2}{R^2}\right) + m_2\right]a$$

$$a = \frac{m_2}{\left[\left(m_1\frac{K^2}{R^2}\right) + m_2\right]}g$$

The expression $\left(\frac{K^2}{R^2}\right)$ for a disc rotating about an axis passing through the centre and perpendicular to the plane is, $\frac{K^2}{R^2} = \frac{1}{2}$. (Ref Table 5.3) Now the expression for acceleration further simplifies as,

$$a = \frac{m_2}{\left[\left(\frac{m_1}{2}\right) + m_2\right]}g\,; \quad a = \frac{2m_2}{\left[m_1 + 2m_2\right]}g$$

substituting the values,

$$a = \frac{2 \times 0.1}{\left[0.5 + 0.2\right]} \times 10 = \frac{0.2}{0.7} \times 10$$

$$a = 2.857\,\mathrm{m\,s^{-2}}$$

5.5.2 Conservation of Angular Momentum

When no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum.

$$\tau = \frac{dL}{dt}$$

If $\tau = 0$ then, $L = \text{constant}$ (5.49)

As the angular momentum is $L = I\omega$, the conservation of angular momentum could further be written for initial and final situations as,

$$I_i\omega_i = I_f\omega_f \text{ (or) } I\omega = \text{constant} \quad (5.50)$$

The above equations say that if I increases ω will decrease and vice-versa to keep the angular momentum constant.

There are several situations where the principle of conservation of angular momentum is applicable. One striking example is an ice dancer as shown in Figure 5.27. The dancer spins slowly when the hands are stretched out and spins faster when the hands are brought close to the body. Stretching of hands away from body increases moment of inertia, thus the angular velocity decreases resulting in slower spin. When the hands are brought close to the body, the moment of inertia decreases, and thus the angular velocity increases resulting in faster spin.

A diver while in air as in Figure 5.28 curls the body close to decrease the moment of inertia, which in turn helps to increase the number of somersaults in air.

Figure 5.28 Conservation of angular momentum for a diver

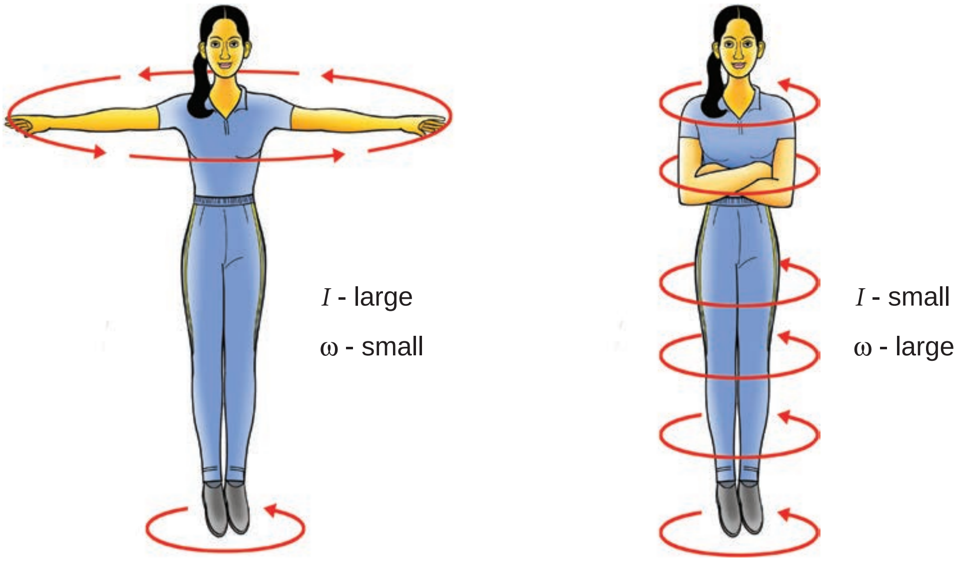


Figure 5.27 Conservation of angular momentum for ice dancer

EXAMPLE 5.19

A jester in a circus is standing with his arms extended on a turn table rotating with angular velocity ω. He brings his arms closer to his body so that his moment of inertia is reduced to one third of the original value. Find his new angular velocity. [Given: There is no external torque on the turn table in the given situation.]

Solution

Let the moment of inertia of the jester with his arms extended be I. As there is no external torque acting on the jester and the turn table, his total angular momentum is conserved. We can write the equation,

$$I_i\omega_i = I_f\omega_f$$

$$I_i\,\omega_i = \frac{1}{3}I_i\,\omega_f \qquad \left(\because I_f = \frac{1}{3}I_i\right)$$

$$\omega_f = 3\omega_i$$

The above result tells that the final angular velocity is three times that of initial angular velocity.

5.5.3 Work done by Torque

Let us consider a rigid body rotating about a fixed axis. Figure 5.29 shows a point P on the body rotating about an axis perpendicular to the plane of the page. A tangential force F is applied on the body.

It produces a small displacement ds on the body. The work done (dw) by the force is,

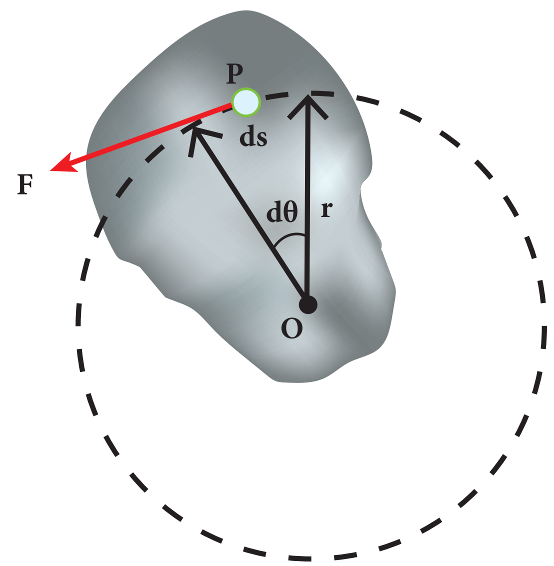


Figure 5.29 Work done by torque

$$dw = F\,ds$$

As the distance ds, the angle of rotation dθ and radius r are related by the expression,

$$ds = r\,d\theta$$

The expression for work done now becomes,

$$dw = F\,ds; \quad dw = F\,r\,d\theta$$

The term (Fr) is the torque τ produced by the force on the body.

$$dw = \tau\,d\theta \qquad (5.51)$$

This expression gives the work done by the external torque τ, which acts on the body rotating about a fixed axis through an angle dθ.

The corresponding expression for work done in translational motion is,

$$dw = F\,ds$$

5.5.4 Kinetic Energy in Rotation

Let us consider a rigid body rotating with angular velocity ω about an axis as shown in Figure 5.30. Every particle of the body will have the same angular velocity ω and different tangential velocities v based on its positions from the axis of rotation.

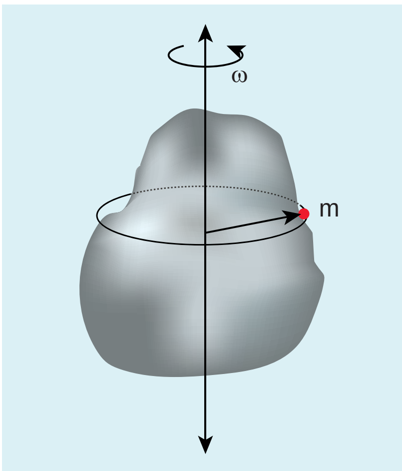


Figure 5.30 Kinetic energy in rotation

Let us choose a particle of mass m_i situated at distance r_i from the axis of rotation. It has a tangential velocity v_i given by the relation, $v_i = r_i\,\omega$. The kinetic energy KE_i of the particle is,

$$KE_i = \frac{1}{2}m_i v_i^2$$

Writing the expression with the angular velocity,

$$KE_i = \frac{1}{2}m_i\left(r_i\omega\right)^2 = \frac{1}{2}\left(m_i r_i^2\right)\omega^2$$

For the kinetic energy of the whole body, which is made up of large number of such particles, the equation is written with summation as,

$$KE = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

where, the term $\sum m_i r_i^2$ is the moment of inertia I of the whole body. $I = \sum m_i r_i^2$

Hence, the expression for KE of the rigid body in rotational motion is,

$$KE = \frac{1}{2}I\omega^2 \qquad (5.52)$$

This is analogous to the expression for kinetic energy in translational motion.

$$KE = \frac{1}{2}Mv^2$$

Relation between rotational kinetic energy and angular momentum

Let a rigid body of moment of inertia I rotate with angular velocity ω.

The angular momentum of a rigid body is, $L = I\omega$

The rotational kinetic energy of the rigid body is, $KE = \frac{1}{2}I\omega^2$

By multiplying the numerator and denominator of the above equation with I, we get a relation between L and KE as,

$$KE = \frac{1}{2}\frac{I^2\omega^2}{I} = \frac{1}{2}\frac{(I\omega)^2}{I}$$

$$KE = \frac{L^2}{2I} \qquad (5.53)$$

EXAMPLE 5.20

Find the rotational kinetic energy of a ring of mass 9 kg and radius 3 m rotating with 240 rpm about an axis passing through its centre and perpendicular to its plane. (rpm is a unit of speed of rotation which means revolutions per minute)

Solution

The rotational kinetic energy is, $KE = \frac{1}{2} I\omega^2$

The moment of inertia of the ring is, $I = MR^2$

$$I = 9 \times 3^2 = 9 \times 9 = 81\,\text{kg m}^2$$

The angular speed of the ring is,

$$\omega = 240\,\text{rpm} = \frac{240 \times 2\pi}{60}\,\text{rad s}^{-1}$$

$$KE = \frac{1}{2} \times 81 \times \left(\frac{240 \times 2\pi}{60}\right)^2 = \frac{1}{2} \times 81 \times (8\pi)^2$$

$$KE = \frac{1}{2} \times 81 \times 64 \times (\pi)^2 = 2592 \times (\pi)^2$$

$$KE \approx 25920\,\text{J} \quad \because (\pi)^2 \approx 10$$

$$KE = 25.920\ \text{kJ}$$

5.5.5 Power Delivered by Torque

Power delivered is the work done per unit time. If we differentiate the expression for work done with respect to time, we get the instantaneous power (P).

$$P = \frac{dw}{dt} = \tau \frac{d\theta}{dt} \quad \because (dw = \tau d\theta)$$

$$P = \tau\omega \tag{5.54}$$

The analogous expression for instantaneous power delivered in translational motion is,

$$P = \vec{F} \cdot \vec{v}$$

5.5.6 Comparison of Translational and Rotational Quantities

Many quantities in rotational motion have expressions similar to that of translational motion. The rotational terms are compared with the translational equivalents in Table 5.4.

Table 5.4 Comparison of Translational and Rotational Quantities

S.No	Translational Motion	Rotational motion about a fixed axis
1	Displacement, x	Angular displacement, θ
2	Time, t	Time, t
3	Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
4	Acceleration, $a = \frac{dv}{dt}$	Angular acceleration, $\alpha = \frac{d\omega}{dt}$
5	Mass, m	Moment of inertia, I
6	Force, F = ma	Torque, τ = I α
7	Linear momentum, p = mv	Angular momentum, L = Iω
8	Impulse, F Δt = Δp	Angular impulse, τΔt = ΔL
9	Work done, w = F s	Work done, w = τ θ
10	Kinetic energy, $KE = \frac{1}{2} m v^2$	Kinetic energy, $KE = \frac{1}{2} I\omega^2$
11	Power, P = F v	Power, P = τω

UNIT-5(XI-Physics_Vol-1).indd 250 06-01-2022 17:48:02

5.6 ROLLING MOTION

The rolling motion is the most commonly observed motion in daily life. The motion of wheel is an example of rolling motion. Round objects like ring, disc, sphere etc. are most suitable for rolling .

Let us study the rolling of a disc on a horizontal surface. Consider a point P on the edge of the disc. While rolling, the point undergoes translational motion along with its centre of mass and rotational motion with respect to its centre of mass.

5.6.1 Combination of Translation and Rotation

We will now see how these translational and rotational motions are related in rolling. If the radius of the rolling object is R, in one full rotation, the centre of mass is displaced by $2\pi R$ (its circumference). One would agree that not only the centre of mass, but all the points on the disc are displaced by the same $2\pi R$ after one full rotation. The only difference is that the centre of mass takes a straight path; but, all the other points undergo a path which has a combination of the translational and rotational motion. Especially the point on the edge undergoes a path of a cycloid as shown in the Figure 5.31.

As the centre of mass takes only a straight line path, its velocity v_{CM} is only translational velocity v_{TRANS} ($v_{CM} = v_{TRANS}$). All the other points have two velocities. One is the translational velocity v_{TRANS}, (which is also the velocity of centre of mass) and the other is the rotational velocity v_{ROT} ($v_{ROT} = r\omega$). Here, r is the distance of the point from the centre of mass and ω is the angular velocity. The rotational velocity v_{ROT} is perpendicular to the instantaneous position vector from the centre of mass as shown in Figure 5.32(a). The resultant of these two velocities is v. This resultant velocity v is perpendicular to the position vector from the point of contact of the rolling object

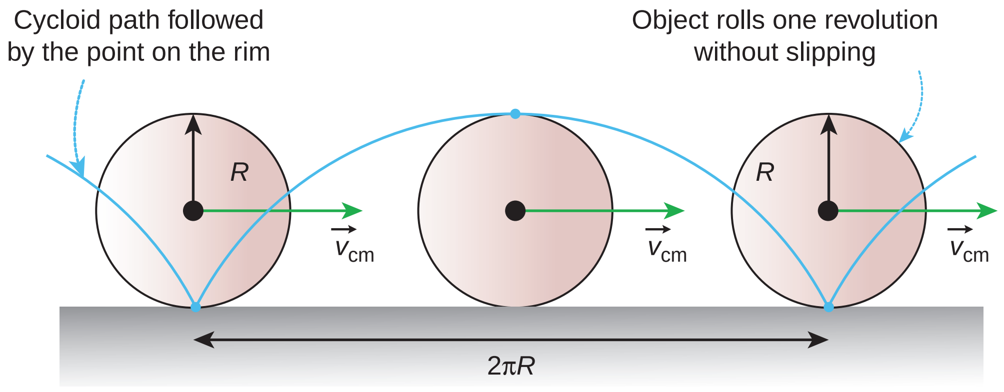
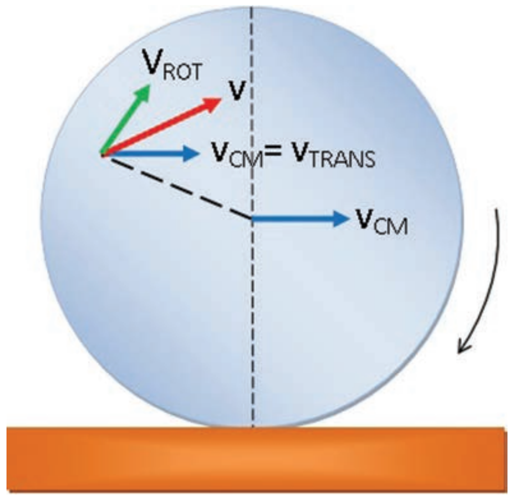
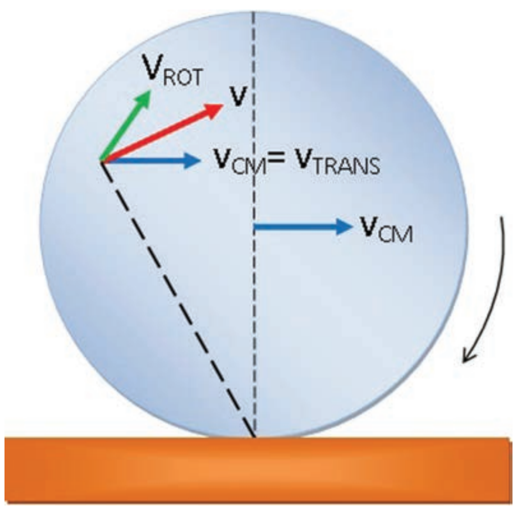


Figure 5.31 Rolling is combination of translation and rotation



(a) with respect to centre of mass



(b) with respect to point of contact

Figure 5.32 Resultant velocity at a point

with the surface on which it is rolling as shown in Figure 5.32(b).

We shall now give importance to the point of contact. In pure rolling, the point of the rolling object which comes in contact with the surface is at momentary rest. This is the case with every point that is on the edge of the rolling object. As the rolling proceeds, all the points on the edge, one by one come in contact with the surface; remain at momentary rest at the time of contact and then take the path of the cycloid as already mentioned.

Hence, we can consider the pure rolling in two different ways.

(i) The combination of translational motion and rotational motion about the centre of mass.

(or)

(ii) The momentary rotational motion about the point of contact.

As the point of contact is at momentary rest in pure rolling, its resultant velocity v is zero (v = 0). For example, in Figure 5.33, at the point of contact, v_{TRANS} is forward (to right) and v_{ROT} is backwards (to the left).

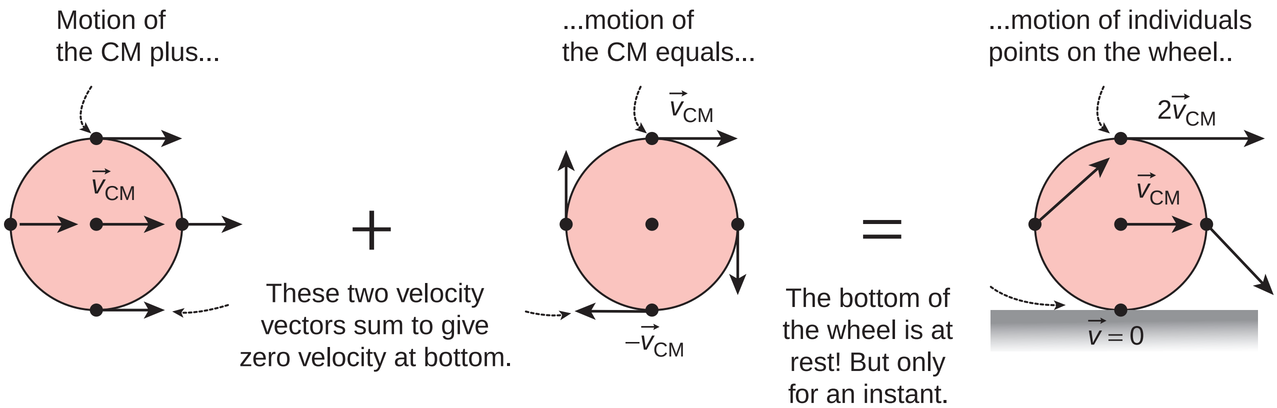


Figure 5.33 In pure rolling, the point of contact is at rest

That implies that, v_{TRANS} and v_{ROT} are equal in magnitude and opposite in direction ($v = v_{TRANS} - v_{ROT} = 0$). Hence, we conclude that in pure rolling, for all the points on the edge, the magnitudes of v_{TRANS} and v_{ROT} are equal ($v_{TRANS} = v_{ROT}$). As $v_{TRANS} = v_{CM}$ and $v_{ROT} = R\omega$, in pure rolling we have,

$$v_{CM} = R\,\omega \qquad (5.55)$$

We should remember the special feature of the equation 5.55. In rotational motion, as per the relation $v = r\omega$, the centre point will not have any velocity as r is zero. But in rolling motion, it suggests that the centre point has a velocity v_{CM} given by equation 5.55.

For the topmost point, the two velocities v_{TRANS} and v_{ROT} are equal in magnitude and in the same direction (to the right). Thus, the resultant velocity v is the sum of these two velocities, $v = v_{TRANS} + v_{ROT}$. In other form, $v = 2\,v_{CM}$ as shown in Figure 5.34.

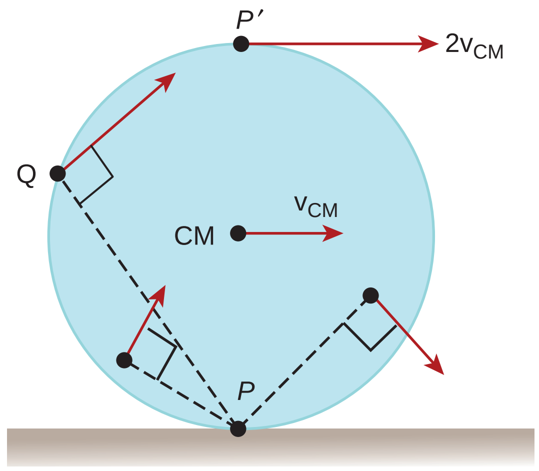
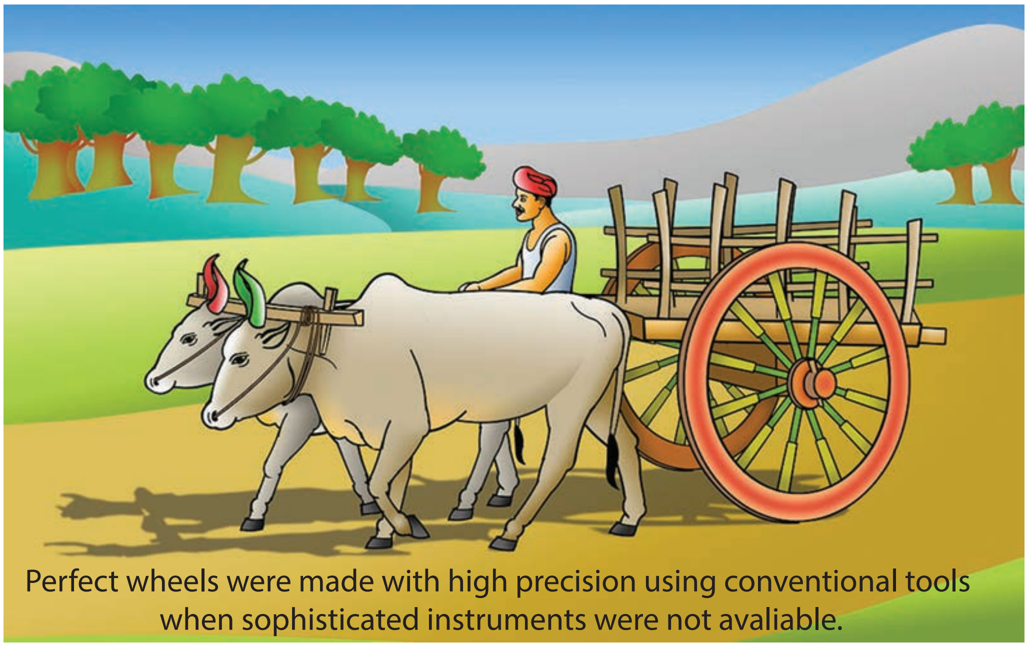


Figure 5.34 Velocity of different point in pure rolling

5.6.2 Slipping and Sliding

When the round object moves, it always tends to roll on any surface which has a coefficient of friction any value greater than zero ($\mu > 0$). The friction that enabling the rolling motion is called rolling friction. In pure rolling, there is no relative motion of the point of contact with the surface. When the rolling object speeds up or slows down,



it must accelerate or decelerate respectively. If this suddenly happens it makes the rolling object to slip or slide.

Sliding

Sliding is the case when $v_{CM} > R\omega$ (or $v_{TRANS} > v_{ROT}$). The translation is more than the rotation. This kind of motion happens when sudden break is applied in a moving vehicles, or when the vehicle enters into a slippery road. In this case, the point of contact has more of v_{TRANS} than v_{ROT}. Hence, it has a resultant velocity v in the forward direction as shown in Figure 5.35. The kinetic frictional force (f_k) opposes the relative motion. Hence, it acts in the opposite direction of the relative velocity. This frictional force reduces the translational velocity and increases the rotational velocity till they become equal and the object sets on pure rolling. Sliding is also referred as forward slipping.

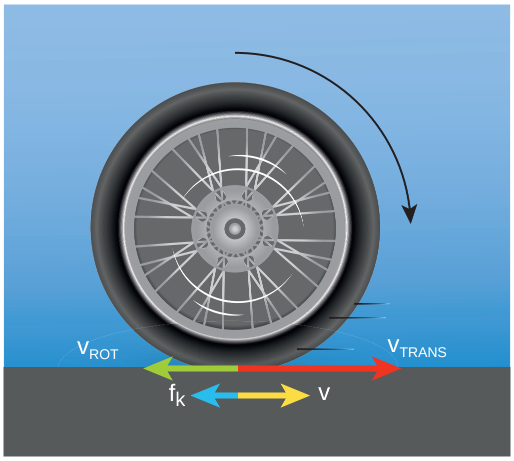


Figure 5.35 Sliding

Slipping

Slipping is the case when $v_{CM} < R\omega$ (or $v_{TRANS} < v_{ROT}$). The rotation is more than the translation. This kind of motion happens when we suddenly start the vehicle from rest or the vehicle is stuck in mud. In this case, the point of contact has more of v_{ROT} than v_{TRANS}. It has a resultant velocity v in the backward direction as shown in Figure 5.36. The kinetic frictional force (f_k) opposes the relative motion. Hence it acts in the opposite direction of the relative velocity. This frictional force reduces the rotational velocity and increases the translational velocity till they become equal and the object sets pure rolling. Slipping is sometimes empahasised as backward slipping.

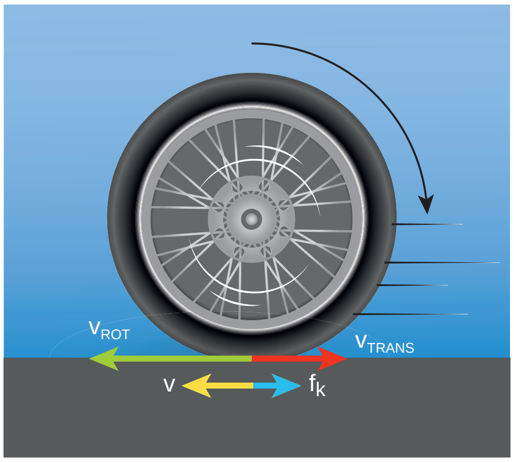


Figure 5.36 Slipping

EXAMPLE 5.21

A rolling wheel has velocity of its centre of mass as 5 m s^{-1}. If its radius is 1.5 m and angular velocity is 3 rad s^{-1}, then check whether it is in pure rolling or not.

Solution

Translational velocity (v_{TRANS}) or velocity of centre of mass, $v_{CM} = 5\ m\ s^{-1}$

The radius is, $R = 1.5$ m and the angular velocity is, $\omega = 3\ rad\ s^{-1}$

Rotational velocity, $v_{ROT} = R\omega$

$$v_{ROT} = 1.5 \times 3$$
$$v_{ROT} = 4.5\ m\ s^{-1}$$

As $v_{CM} > R\omega$ (or) $v_{TRANS} > R\omega$, It is not in pure rolling, but sliding.

5.6.3 Kinetic Energy in Pure Rolling

In genreal pure rolling is the combination of translational and rotational motion, we can write the total kinetic energy (KE) as the sum of kinetic energy due to translational motion (KE_{TRANS}) and kinetic energy due to rotational motion (KE_{ROT}).

$$KE = KE_{TRANS} + KE_{ROT} \qquad (5.56)$$

If the mass of the rolling object is M, the velocity of centre of mass is v_{CM}, its moment of inertia about centre of mass is I_{CM} and angular velocity is ω, then

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 \qquad (5.57)$$

With centre of mass as reference:

The moment of inertia (I_{CM}) of a rolling object about the centre of mass is, $I_{CM} = MK^2$ and $v_{CM} = R\omega$. Here, K is radius of gyration.

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\left(MK^2\right)\frac{v_{CM}^2}{R^2}$$
$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right) \qquad (5.58)$$
$$KE = \frac{1}{2}Mv_{CM}^2\left(1 + \frac{K^2}{R^2}\right) \qquad (5.59)$$

With point of contact as reference:

We can also arrive at the same expression by taking the momentary rotation happening with respect to the point of contact (another approach to rolling). If we take the point of contact as O, then,

$$KE = \frac{1}{2}I_o\omega^2$$

Here, I_o is the moment of inertia of the object about the point of contact. By parallel axis theorem, $I_o = I_{CM} + MR^2$. Further we can write, $I_o = MK^2 + MR^2$. With $v_{CM} = R\omega$ or $\omega = \frac{v_{CM}}{R}$

$$KE = \frac{1}{2}\left(MK^2 + MR^2\right)\frac{v_{CM}^2}{R^2}$$
$$KE = \frac{1}{2}Mv_{CM}^2\left(1 + \frac{K^2}{R^2}\right) \qquad (5.60)$$

As the two equations 5.59 and 5.60 are the same, it is once again confirmed that the

UNIT-5(XI-Physics_Vol-1).indd 255 06-01-2022 17:48:07

pure rolling problems could be solved by considering the motion as any one of the following two cases.

(i) The combination of translational motion and rotational motion about the centre of mass. (or)

(ii) The momentary rotational motion about the point of contact.

EXAMPLE 5.22

A solid sphere is undergoing pure rolling. What is the ratio of its translational kinetic energy to rotational kinetic energy?

Solution

The expression for total kinetic energy in pure rolling is,

$$KE = KE_{TRANS} + KE_{ROT}$$

For any object the total kinetic energy as per equation 5.58 and 5.59 is,

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

$$KE = \frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right)$$

Then,

$$\frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right) = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

The above equation suggests that in pure rolling the ratio of total kinetic energy, translational kinetic energy and rotational kinetic energy is given as,

$$KE : KE_{TRANS} : KE_{ROT} :: \left(1+\frac{K^2}{R^2}\right) : 1 : \left(\frac{K^2}{R^2}\right)$$

$$\text{Now, } KE_{TRANS} : KE_{ROT} :: 1 : \left(\frac{K^2}{R^2}\right)$$

$$\text{For a solid sphere, } \frac{K^2}{R^2} = \frac{2}{5}$$

$$\text{Then, } KE_{TRANS} : KE_{ROT} :: 1 : \frac{2}{5} \quad \text{or}$$

$$KE_{TRANS} : KE_{ROT} :: 5 : 2$$

5.6.4 Rolling on Inclined Plane

Let us assume a round object of mass m and radius R is rolling down an inclined plane without slipping as shown in Figure 5.37. There are two forces acting on the object along the inclined plane. One is the component of gravitational force (mg sinθ) and the other is the static frictional force (f). The other component of gravitation force (mg cosθ) is cancelled by the normal force (N) exerted by the plane. As the motion is happening along the incline, we shall write the equation for motion from the free body diagram (*FBD*) of the object.

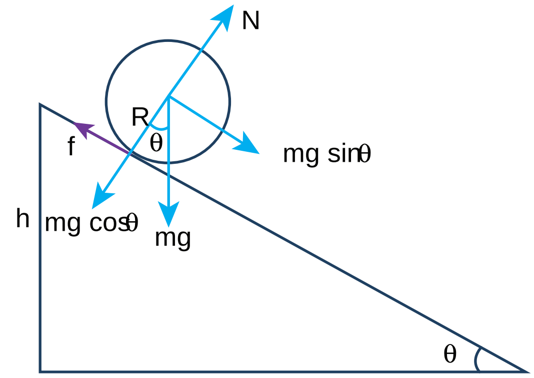


Figure 5.37 Rolling on inclined plane

For translational motion, mg $\sin\theta$ is the supporting force and f is the opposing force,

$$mg\sin\theta - f = ma \qquad (5.61)$$

For rotational motion, let us take the torque with respect to the centre of the object. Then mg $\sin\theta$ cannot cause torque as it passes through it but the frictional force f can set torque of Rf.

$$Rf = I\alpha$$

By using the relation, $a = r\,\alpha$, and moment of inertia $I = mK^2$, we get,

$$Rf = mK^2\frac{a}{R}; \quad f = ma\left(\frac{K^2}{R^2}\right)$$

Now equation (5.61) becomes,

$$mg\sin\theta - ma\left(\frac{K^2}{R^2}\right) = ma$$

$$mg\sin\theta = ma + ma\left(\frac{K^2}{R^2}\right)$$

$$a\left(1+\frac{K^2}{R^2}\right) = g\sin\theta$$

After rewriting it for acceleration, we get,

$$a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} \qquad (5.62)$$

We can also find the expression for final velocity of the rolling object by using third equation of motion for the inclined plane. $v^2 = u^2 + 2as$. If the body starts rolling from rest, $u = 0$. When h is the vertical height of the incline, the length of the incline s is, $s = \frac{h}{\sin\theta}$

$$v^2 = 2\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}\left(\frac{h}{\sin\theta}\right) = \frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}$$

By taking square root,

$$v = \sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}} \qquad (5.63)$$

The time taken for rolling down the incline could also be written from first equation of motion as, $v = u + at$. For the object which starts rolling from rest, $u = 0$. Then,

$$t = \frac{v}{a}$$

$$t = \left(\sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}\right)\left(\frac{\left(1+\frac{K^2}{R^2}\right)}{g\sin\theta}\right)$$

$$t = \sqrt{\frac{2h\left(1+\frac{K^2}{R^2}\right)}{g\sin^2\theta}} \qquad (5.64)$$

The equation suggests that for a given incline, the object with the least value of radius of gyration K will reach the bottom of the incline first.

EXAMPLE 5.23

Four round objects namely a ring, a disc, a hollow sphere and a solid sphere with same radius R start to roll down an incline at the same time. Find out which object will reach the bottom first.

Solution

For all the four objects namely the ring, disc, hollow sphere and solid sphere, the radii of gyration K are R, $\sqrt{\frac{1}{2}}R$, $\sqrt{\frac{2}{3}}R$, $\sqrt{\frac{2}{5}}R$ (ref Table (5.3)). With numerical values the radius of gyration K are 1R, 0.707R, 0.816R, 0.632R respectively. The expression for time taken for rolling has the radius of gyration K in the numerator as per equation 5.63

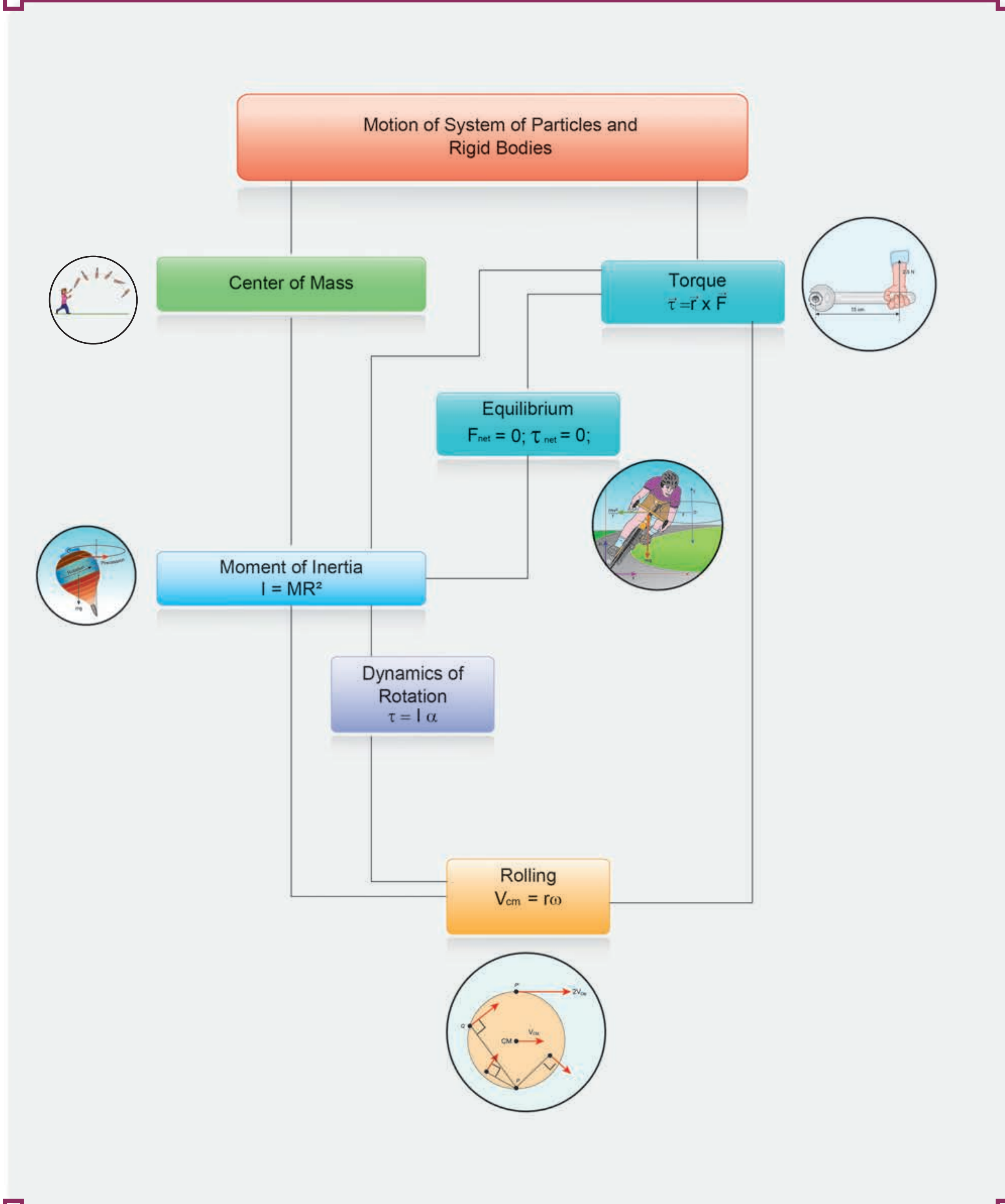
$$t = \sqrt{\frac{2h\left(1+\frac{K^2}{R^2}\right)}{g\sin^2\theta}}$$

The one with least value of radius of gyration K will take the shortest time to reach the bottom of the inclined plane. The order of objects reaching the bottom is first, solid sphere; second, disc; third, hollow sphere and last, ring.

SUMMARY

- A rigid body is the one in which the distances between different particles remain constant.
- For regular shaped bodies with uniform mass distribution, centre of mass always lies at the geometrical centre.
- Net torque produces turning motion in rigid object.
- A rigid body is in translational equilibrium if the total external force on it is zero. It is in rotational equilibrium if the total external torque on it is zero.
- The centre of gravity of an extended body is that point where the total gravitational torque on the body is zero.
- If the external torque acting on the body is zero, the component of angular momentum along the axis of rotation is constant.
- There are rotational equivalents for all the translational quantities.
- Rolling motion is the combination of translational and rotational motions.
- Rolling can also be treated as the momentary rotation about the point of contact.
- In pure rolling, the total kinetic energy is the sum of kinetic energies of translational and rotational motions.
- In sliding the translational motion is more than rotational motion.
- In slipping the rotational motion is more than translational motion.

CONCEPT MAP


Motion of System of Particles and Rigid Bodies
Center of Mass
Torque
$\vec{\tau} = \vec{r} \times \vec{F}$
Equilibrium
$F_{net} = 0; \tau_{net} = 0;$
Moment of Inertia
$I = MR^2$
Dynamics of Rotation
$\tau = I\alpha$
Rolling
$V_{cm} = r\omega$

EVALUATION

I. Multiple Choice Questions

1. The centre of mass of a system of particles does not depend upon,

(a) position of particles

(b) relative distance between particles

(c) masses of particles

(d) force acting on particle

[AIPMT 1997, AIEEE 2004]

2. A couple produces,

(a) pure rotation

(b) pure translation

(c) rotation and translation

(d) no motion

[AIPMT 1997]

3. A particle is moving with a constant velocity along a line parallel to positive X-axis. The magnitude of its angular momentum with respect to the origin is,

(a) zero

(b) increasing with x

(c) decreasing with x

(d) remaining constant

[IIT 2002]

4. A rope is wound around a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force 30 N?

(a) 0.25 rad s^{-2} (b) 25 rad s^{-2}

(c) 5 m s^{-2} (d) 25 m s^{-2}.

[NEET 2017]

5. A closed cylindrical container is partially filled with water. As the container rotates in a horizontal plane about a perpendicular bisector, its moment of inertia,

(a) increases (b) decreases

(c) remains constant

(d) depends on direction of rotation.

[IIT 1998]

6. A rigid body rotates with an angular momentum L. If its kinetic energy is halved, the angular momentum becomes,

(a) L (b) L/2

(c) 2L (d) $L/\sqrt{2}$

[AFMC 1998, AIPMT 2015]

7. A particle undergoes uniform circular motion. The angular momentum of the particle remain conserved about,

(a) the centre point of the circle.

(b) the point on the circumference of the circle.

(c) any point inside the circle.

(d) any point outside the circle.

[IIT 2003]

8. When a mass is rotating in a plane about a fixed point, its angular momentum is directed along,

(a) a line perpendicular to the plane of rotation

(b) the line making an angle of 45° to the plane of rotation

(c) the radius

(d) tangent to the path

[AIPMT 2012]

9. Two discs of same moment of inertia rotating about their regular axis passing through centre and perpendicular to

UNIT-5(XI-Physics_Vol-1).indd 260 06-01-2022 17:48:17

the plane of disc with angular velocities ω_1 and ω_2. They are brought in to contact face to face coinciding the axis of rotation. The expression for loss of energy during this process is,

(a) $\frac{1}{4}I(\omega_1-\omega_2)^2$ (b) $I(\omega_1-\omega_2)^2$

(c) $\frac{1}{8}I(\omega_1-\omega_2)^2$ (d) $\frac{1}{2}I(\omega_1-\omega_2)^2$

[NEET 2017]

10. A disc of moment of inertia I_a is rotating in a horizontal plane about its symmetry axis with a constant angular speed ω. Another discinitially at rest of moment of inertia I_b is dropped coaxially on to the rotating disc. Then, both the discs rotate with same constant angular speed. The loss of kinetic energy due to friction in this process is,

(a) $\frac{1}{2}\frac{I_b^2}{(I_a+I_b)}\omega^2$

(b) $\frac{I_b^2}{(I_a+I_b)}\omega^2$

(c) $\frac{(I_b-I_a)^2}{(I_a+I_b)}\omega^2$

(d) $\frac{1}{2}\frac{I_bI_b}{(I_a+I_b)}\omega^2$ [AIPMT 2001]

11. The ratio of the acceleration for a solid sphere (mass m and radius R) rolling down an incline of angle θ without slipping and slipping down the incline without rolling is,

(a) 5:7 (b)2:3 (c) 2:5 (d) 7:5

[AIPMT 2014]

12. From a disc of radius R a mass M, a circular hole of diameter R, whose rim passes through the centre is cut. What is the moment of inertia of the remaining part of the disc about a perpendicular axis passing through it

(a) $15MR^2/32$ (b) $13MR^2/32$

(c) $11MR^2/32$ (d) $9MR^2/32$

[NEET 2016]

13. The speed of a solid sphere after rolling down from rest without sliding on an inclined plane of vertical height h is,

(a) $\sqrt{\frac{4}{3}gh}$ (b) $\sqrt{\frac{10}{7}gh}$

(c) $\sqrt{2gh}$ (d) $\sqrt{\frac{1}{2}gh}$

14. The speed of the centre of a wheel rolling on a horizontal surface is v_o. A point on the rim in level with the centre will be moving at a speed of speed of,

(a) zero (b) v_o

(c) $\sqrt{2}v_o$ (d) $2v_o$

[PMT 1992, PMT 2003, IIT 2004]

15. A round object of mass M and radius R rolls down without slipping along an inclined plane. The frictional force,

(a) dissipates kinetic energy as heat.

(b) decreases the rotational motion.

(c) decreases the rotational and transnational motion

(d) converts transnational energy into rotational energy

[PMT 2005]

Answers:

1) d	2) a	3) d	4) b
5) a	6) d	7) a	8) a
9) a	10) d	11) a	12) b
13) b	14) c	15) d	

II. Short Answer Questions

1. Define centre of mass.
2. Find out the centre of mass for the given geometrical structures.
   a) Equilateral triangle
   b) Cylinder c) Square
3. Define torque and mention its unit.
4. What are the conditions in which force can not produce torque?
5. Give any two examples of torque in day-to-day life.
6. What is the relation between torque and angular momentum?
7. What is equilibrium?
8. How do you distinguish between stable and unstable equilibrium?
9. Define couple.
10. State principle of moments.
11. Define centre of gravity.
12. Mention any two physical significance of moment of inertia
13. What is radius of gyration?
14. State conservation of angular momentum.
15. What are the rotational equivalents for the physical quantities, (i) mass and (ii) force?
16. What is the condition for pure rolling?
17. What is the difference between sliding and slipping?

III. Long Answer Questions

1. Explain the types of equilibrium with suitable examples.
2. Explain the method to find the centre of gravity of a irregularly shaped lamina.
3. Explain why a cyclist bends while negotiating a curve road? Arrive at the expression for angle of bending for a given velocity.
4. Derive the expression for moment of inertia of a rod about its centre and perpendicular to the rod.
5. Derive the expression for moment of inertia of a uniform ring about an axis passing through the centre and perpendicular to the plane.
6. Derive the expression for moment of inertia of a uniform disc about an axis passing through the centre and perpendicular to the plane.
7. Discuss conservation of angular momentum with example.
8. State and prove parallel axis theorem.
9. State and prove perpendicular axis theorem.
10. Discuss rolling on inclined plane and arrive at the expression for the acceleration.

IV. Numerical Problems

1. A uniform disc of mass 100g has a diameter of 10 cm. Calculate the total energy of the disc when rolling along a horizontal table with a velocity of 20 cms^{-1}. (take the surface of table as reference)

   Ans: 0.005 J

2. A particle of mass 5 units is moving with a uniform speed of $v = 3\sqrt{2}$ units in the XOY plane along the line y = x + 4. Find the magnitude of angular momentum.

   Ans: 60 units

3. A fly wheel rotates with a uniform angular acceleration. If its angular velocity increases from 20π rad/s to 40π rad/s in 10 seconds. Find the number of rotations in that period.

Ans: 150 rotations

4. A uniform rod of mass m and length ℓ makes a constant angle θ with an axis of rotation which passes through one end of the rod. Find the moment of inertia about this axis.

Ans: $\frac{1}{3}M\ell^2 \sin^2\theta$

5. Two particles P and Q of mass 1kg and 3 kg respectively start moving towards each other from rest under mutual attraction. What is the velocity of their centre of mass?

Ans: Zero

6. Find the moment of inertia of a hydrogen molecule about an axis passing through its centre of mass and perpendicular to the inter-atomic axis. Given: mass of hydrogen atom 1.7×10^{-27} kg and inter atomic distance is equal to 4×10^{-10}m.

Ans: 1.36×10^{-46} kg m^2

BOOKS FOR REFERENCE

1. Michael Nelkon and Philip Parker, Advanced Level Physics, 7th Edition, CBS Publishers & Distributers Pvt. Ltd, (2006).

2. David Halliday, Robert Resnick and Jearl Walker,Fundamentals of Physics, 6th Edition, John Wiley & Sons Inc., (2004).

3. H.C. Verma,Concepts of Physics [Part 1], 1st Edition, BharathiBhawan Publishers & Distributers Pvt. Ltd., (2008).

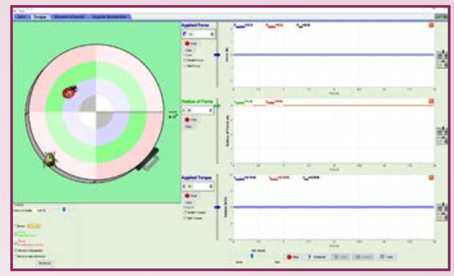
4. Igor Irodov, Problems in General Physics, 3rd Edition, Mir Publishers, Mascow, (2006).

5. Roger A. Freedman, Hugh D. Young, Sears and Zemansky's University Physics: Mechanics, 12th Edition, Pearson, (2011).

ICT CORNER

Moment of inertia

Which is harder to rotate:
Circular ring or Circular disc?



STEPS:

- Open the browser and type the given URL to open the PhET simulation on Torque. Click the picture link or the download button. Once downloaded, click ok to open the java applet.
- Set platform mass 0.1 kg, Outer radius 4 m. (Keep inner radius = 0). Now it is a circular disc. Click the button 'go' to get the value of moment of inertia.
- Adjust the values of mass and radius and then observe how moment of inertia changes in the middle graph.
- Keep the inner and outer radius same (say R = r = 4 m.) and mass 0.1 kg. Now it becomes circular ring. Click the button 'go' to start the calculation.
- Observe the moment of inertia from the middle graph. Compare the moment of inertia of a circular disc and circular ring with same mass and radius.

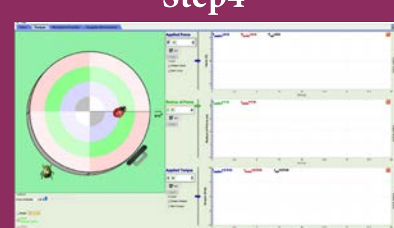
Hint: If moment of inertia is relatively large, it is very difficult to accelerate in angular direction.

Step1

Step2

Step3

Step4



PhET simulation's URL:

https://phet.colorado.edu/en/simulation/torque

* Pictures are indicative only.

*If browser requires, allow Flash Player or Java Script to load the page.

DY5RT

APPENDIX 1

SOLVED EXAMPLE UNIT-1

1. Find the dimensions of a and b in the formula $[P+\frac{a}{V^2}][V-b]=RT$ where P is pressure and V is the volume of the gas

Solution:

By the principle of homogeneity, a/V^2 is of the dimensions of pressure and b is of the dimensions of volume.

$[a] = [\text{pressure}]\,[V^2] = [ML^{-1}T^{-2}]\,[L^6]$
$= [ML^5T^{-2}]$

$[b] = [V] = [L^3]$

2. Show that $(P^{-5/6}\rho^{1/2}E^{1/3})$ is of the dimension of time. Here P is the pressure, ρ is the density and E is the energy of a bubble)

Solution:

Dimension of Pressure $= [ML^{-1}T^{-2}]$

Dimension of density $= [ML^{-3}]$

Dimension of Energy $= [ML^2T^{-2}]$

By substituting in the given equation,

$$= [ML^{-1}T^{-2}]^{-5/6}[ML^{-3}]^{1/2}[ML^2T^{-2}]^{1/3}$$
$$= M^{-5/6+1/2+1/3}L^{5/6-3/2+2/3}T^{5/3-2/3}$$
$$= M^0L^0T^1 = [T]$$

3. Find the dimensions of mass in terms of Energy, length and time

Solution:

Let the dimensions of Energy, Length and Time be $[E],[L],[T]$ respectively.

We know that Force = mass x acceleration

$$\text{Mass} = \frac{Force}{acceleration}$$
$$= \frac{Workdone\ (or)\ Energy}{acceleration \times displacement}$$
$$[m] = \frac{Energy}{[acceleration][displacement]}$$
$$= \frac{[E]}{[LT^{-2}][L]} = \frac{[E]}{L^2T^{-2}} = [EL^{-2}T^2]$$

4. A physical quantity Q is found to depend on quantities x,y,z obeying

relation Q = $\frac{x^2 y^3}{z^1}$. The percentage errors in x, y and z are 2%, 3% and 1% respectively. Find the percentage error in Q.

Solution:

$$\text{Let, } Q = \frac{x^2 y^3}{z}.$$

$$\text{It is given, } \frac{\Delta x}{x} = 2\% \quad \frac{\Delta y}{y} = 3\% \quad \frac{\Delta z}{z} = 1\%$$

$$\frac{\Delta Q}{Q} = 2\left(\frac{\Delta x}{x}\right) + 3\left(\frac{\Delta y}{y}\right) + 1\left(\frac{\Delta z}{z}\right)$$

$$= 2(2\%) + 3\,(3\%) + 1(1\%)$$

$$\frac{\Delta Q}{Q} = 4\% + 9\% + 1\% = 14\%$$

5. The mass and volume of a body are found to be 4±.03 kg and 5±.01 m^3 respectively. Then find the maximum possible percentage error in density.

Solution:

$$\text{Mass m} = 4 \pm 0.03 \text{ kg } (m + \Delta m)$$

$$\text{Volume V} = 5 \pm .01 \text{ m}^3 \ (V + \Delta V)$$

$$\text{Density} = ?$$

$$\text{Error in mass } = \frac{\Delta m}{m} = \frac{0.03}{4} \times 100$$

$$= 0.75\%$$

$$\text{Error in volume } = \frac{\Delta V}{V} = \frac{0.01}{5} \times 100$$

$$= 0.2\%$$

$$\text{Density} = \frac{mass}{volume}.$$

Error in density = error in mass + error in volume

$$= 0.75\% + 0.2\% = 0.95\%$$

6. Using a Vernier Callipers, the length of a cylinder in different measurements is found to be 2.36 cm, 2.27 cm, 2.26 cm, 2.28 cm, 2.31 cm, 2.28 cm and 2.29 cm. Find the mean value, absolute error, the relative error and the percentage error of the cylinder.

Solution:

The given readings are 2.36 cm, 2.27 cm, 2.26 cm, 2.28 cm, 2.31 cm, 2.28 cm and 2.29 cm

The Mean value $\bar{l}$ =

$$\frac{2.36 + 2.27 + 2.26 + 2.28 + 2.31 + 2.28 + 2.29}{7}$$

$$\frac{16.05}{7} = 2.29 \text{ cm}$$

Absolute errors in the measurement are,

$$|\Delta l_1| = |2.29 - 2.36| = 0.07$$
$$|\Delta l_2| = |2.29 - 2.27| = 0.02$$
$$|\Delta l_3| = |2.29 - 2.26| = 0.03$$
$$|\Delta l_4| = |2.29 - 2.28| = 0.01$$
$$|\Delta l_5| = |2.29 - 2.31| = 0.02$$
$$|\Delta l_6| = |2.29 - 2.28| = 0.01$$
$$|\Delta l_7| = |2.29 - 2.29| = 0.00$$

Mean Absolute error

Δl_{mean} =

$$\frac{0.07 + 0.02 + 0.03 + 0.01 + 0.02 + 0.01 + 0.00}{7}$$

$$= \frac{.16}{7} = .02$$

Relative error

$$= \frac{\Delta lmean}{\bar{l}} = \pm \frac{.02}{2.29} = \pm 8.7 \times 10^{-3}$$

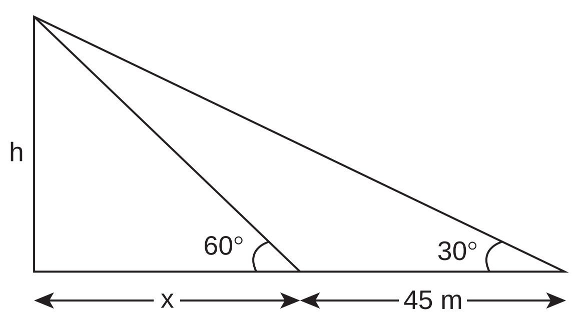
$$\text{Percentage error} = \pm 8.7 \times 10^{-3} \times 100 =$$

$$0.87\% \times 100 = \pm(8.7 \times 10^{-1}) = 0.9\%$$

7. The shadow of a pole standing on a level ground is found to be 45 m longer when the sun's altitude is 30° than when it was 60°. Determine the height of the pole. [Given $\sqrt{3} = 1.73$]

Solution:

Let the height of the pole be h.



$$\text{Solution } \frac{x+45}{h} = \cot 30° \Rightarrow h = \frac{x+45}{\cot 30°}$$

$$\frac{x}{h} = \cot 60° \Rightarrow x = h\cot 60°$$

Substituting the values of x in the above equation

$$h = \frac{h\cot 60° + 45}{\cot 30°}$$

$$h \cot 30° = h \cot 60° + 45$$

$$h(\cot 30° - \cot 60°) = 45$$

$$h = \frac{45}{\cot 30° - \cot 60°} = \frac{45}{\sqrt{3} - \frac{1}{\sqrt{3}}} = 38.97\,\text{m}$$

8. Calculate the number of times a human heart beats in the life of 100 years old man. Time of one heart beat = 0.8s.

Solution:

Life of the man = 100 years

100 years includes 76 normal years and 24 leap years

Total no of days = 76 × 365 + 24 × 366 = 36524 days

Number of seconds = 36524 × 24 × 3600 = 3.155 x 10^9 second

Number of heart beats =

$$\frac{\textit{Total no of Seconds}}{\textit{Time period of heart beat}}$$

$$= \frac{3.155 \times 10^9}{0.8s} = 3.94 \times 10^9$$

9. The parallax of a heavenly body measured from two points diametrically opposite on equator of earth is 2′. Calculate the distance of the heavenly body. [Given radius of the earth = 6400km] [1″ = 4.85 x 10^{-6} rad]

Solution:

Angle θ = 2^1 = 2 x 60″ = 120″ = 120 × 4.85 × 10^{-6} rad

$$\theta = 5.82 \times 10^{-4} \text{ rad;}$$

The distance of heavenly body

$$D = \frac{d}{\theta} = \frac{12800 \times 10^3}{5.82 \times 10^{-4}}$$

$$D = 2.19 \times 10^{10}\,\text{m}$$

10. Convert a velocity of 72 kmh^{-1} into $m\,s^{-1}$ with the help of dimensional analysis.

Solution:

$$n_1 = 72\ kmh^{-1} \quad n_2 = ?\ m\,s^{-1}$$
$$L_1 = 1Km \quad L_2 = 1m$$
$$T_1 = 1h \quad T_2 = 1s$$

$$n_2 = n_1\left[\frac{L_1}{L_2}\right]^a\left[\frac{T_1}{T_2}\right]^b$$

The dimensional formula for velocity is $[L\,T^{-1}]$

$$a = 1\ b = -1$$

$$n_2 = 72\left[\frac{1Km}{1m}\right]^1\left[\frac{1h}{1s}\right]^{-1}$$

$$n_2 = 72\left[\frac{1000m}{1m}\right]^1\left[\frac{3600s}{1s}\right]^{-1}$$

$$= 72 \times 1000 \times 1/3600 = 20m\,s^{-1}]$$

$$72\ kmh^{-1} = 20\ m\,s^{-1}$$

11. Check the correctness of the following equation using dimensional analysis. Make a comment on it.

$S = ut + 1/4\ at^2$ where s is the displacement, u is the initial velocity, t is the time and a is the acceleration produced.

Solution:

Dimension for distance s = [L]

Dimension for initial velocity

$$v = [LT^{-1}]$$

Dimension for time t = [T]

Dimension for acceleration

$$a = [LT^{-2}]$$

According to the principle of homogeneity,

Dimensions of LHS = Dimensions of RHS

Substituting the dimensions in the given formula

$S = ut + 1/4\ at^2$, $\frac{1}{4}$ is a number. It has no dimensions

$$[L] = [LT^{-1}]\ [T^1]+[LT^{-2}]\ [T^2]$$
$$[L] = [L] + [L]$$

As the dimensional formula of LHS is same as that of RHS, the equation is dimensionally correct.

Comment:

But actually it is a ***wrong*** equation. We know that the equation of motion is $s = ut + 1/2\ at^2$

So, a dimensionally correct equation ***need not*** be the true (or) actual equation

But a true equation is ***always*** dimensionally correct.

12. Round - off the following numbers as indicated.

a) 17.234 to 3 digits

b) 3.996×10^5 to 3 digits

c) 3.6925×10^{-3} to 2 digits

d) 124783 to 5 digits.

Solution:

a) 17.2 b) 4.00×10^5

c) 3.7×10^{-3} d) 124780

13. Solve the following with regard to significant figures.

a) $\sqrt{4.5 - 3.31}$

b) $5.9 \times 10^5 - 2.3 \times 10^4$

c) 7.18 + 4.3
d) 6.5 + .0136

Solution:

a) Among the two, the least number of significant after decimal is one

$$\sqrt{4.5-3.31} = \sqrt{1.19} = 1.09$$

b) The number of minimum significant figures is 2

$$5.9\times10^{5}-2.3\times10^{4}$$
$$= 5.9\times10^{5}-0.23\times10^{5}$$
$$= 5.67\times10^{5} = 5.7\times10^{5}$$

c) The lowest least number of significant digit after decimal is one

7.18 + 4.3 = 11.48 Rounding off we get 11.5

d) The lowest least number of significant digit after decimal is one

$$6.5 + .0136 = 6.5136 = 6.5$$

14. Arrive at Einstein's mass-energy relation by dimensional method ($E = mc^2$)

Solution:

Let us assume that the Energy E depends on mass m and velocity of light c.

$$E \propto m^a c^b$$

$E = km^a c^b$ where K a constant

Dimensions of E = $[ML^2T^{-2}]$

Dimensions of m = [M]

Dimensions of c = $[LT^{-1}]$

Substituting the values in the above equation

$$[ML^2T^{-2}] = K\,[M]^a[LT^{-1}]^b$$

By equating the dimensions,

$$a = 1$$
$$b = 2$$
$$-b = -2$$
$$E = k.mc^2$$

The value of constant k = 1

$E = mc^2$ This is Einstein's mass energy relation

15. The velocity of a body is given by the equation $v = b/t + ct^2+dt^3$. Find the dimensional formula for b.

Solution:

(b/t) should have the dimensions of velocity
b has the dimensions of (velocity x time)

$$[b] = [LT^{-1}][T] = [L] = [M^0L^1T^0]$$

16. The initial and final temperatures of a liquid in a container are observed to be 75.4 ±0.5°C and 56.8 ±0.2°C. Find the fall in the temperature of the liquid.

Solution:

$t_1 = (75.4 \pm 0.5)°C$
$t_2 = (56.8 \pm 0.2)°C$
Fall in temperature = (75.4 ±0.5°C) – (56.8 ±0.2°C)

$$t = (18.6 \pm 0.7)°C$$

17. Two resistors of resistances R_1 = 150 ± 2 Ohm and $R_2 = 220 \ \pm 6$ Ohm are connected in parallel combination. Calculate the equivalent resistance.

Hint: $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$

Solution:

The equivalent resistance of a parallel combination

$$R' = \frac{R_1 R_2}{R_1 + R_2}$$

$$= \frac{150 \times 220}{150 + 220} = \frac{33000}{370} = 89.1\ Ohm$$

We know that, $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{\Delta R'}{(R')^2} = \frac{\Delta R_1}{R_1^{\ 2}} + \frac{\Delta R_2}{R_2^{\ 2}}$$

$$\Delta R' = (R')^2 \frac{\Delta R_1}{R_1^{\ 2}} + (R')^2 \frac{\Delta R_2}{R_2^{\ 2}}$$

$$= \left(\frac{R'}{R_1}\right)^2 \Delta R_1 + \left(\frac{R'}{R_2}\right)^2 \Delta R_2$$

Substituting the value,

$$\Delta R' = \left[\frac{89.1}{150}\right]^2 \times 2 + \left[\frac{89.1}{220}\right]^2 \times 6$$

$$= 0.070 + 0.098 = 0.168$$

R′ = 89.1 ±0.168 Ohm.

18. A capacitor of capacitance C = $3.0 \pm 0.1 \mu F$ is charged to a voltage of V = $18 \pm 0.4 Volt$. Calculate the charge Q [Use Q= CV]

Solution:

$(C + \Delta C) = (3.0 \pm 0.1)\ \mu f$

$(V + \Delta V) = (18 \pm 0.4)\ V$

$Q = CV$

$$Q = 3.0 \times 10^{-6} \times 18 = 54 \times 10^{-6} \text{ coulomb}$$

$$\text{Error in C} = \frac{\Delta C}{C} \times 100$$

$$= \frac{0.1}{3} \times 100 = 3.3\%$$

$$\text{Error in V} = \frac{\Delta V}{V} \times 100$$

$$= \frac{0.4}{18} \times 100 = 2.2\%$$

Error in Q = Error in C + Error in V

= 3.3% + 2.2% =5.5%

$\therefore$ Charge $Q = (54 \times 10^{-6} \pm 5.5\%)$ coulomb

SOLVED EXAMPLE UNIT-2

1. The position vector for a particle is represented be $\vec{r} = 3t^2\hat{i} + 5t\hat{j} + 6\hat{k}$, find the velocity and speed of the particle at t = 3 sec?

Solution:

$$\vec{v} = \frac{d\vec{r}}{dt} = 6t\hat{i} + 5\hat{j}.$$

The velocity at any time '*t*' is given by $\vec{v} = 6t\hat{i} + 5\hat{j}$.

The magnitude of velocity is speed. The speed at any time '*t*' is then given by

$$\text{Speed} = \sqrt{(6t)^2 + 5^2} = \sqrt{36t^2 + 25}$$

Now the velocity at t = 3sec is given by

$$\vec{v} = 6(3)\hat{i} + 5\hat{j} = 18\hat{i} + 5\hat{j}.$$

and speed at t = 3 sec, is given by

$$\text{speed} = \sqrt{349}\ ms^{-1}$$

2. A gun is fired from a place which is at distance 1.2 km from a hill. The echo of the sound is heard back at the same place of firing after 8 second. Find the speed of sound.

Solution:

The echo will be heard when the sound reaches back at the place of firing. So, the total distance travelled by sound is 2 × 1.2 km = 2.4 km = 2400 m.

$$\text{speed} = \frac{2400\text{ m}}{8s} = 300\text{ m s}^{-1}$$

3. A train 100 m long is moving with a speed of 60 kmh^{-1}. In how many seconds will it cross a bridge of 1 km long?

Solution:

Total distance to be covered = 1 km + 100 m = 1100 m (including both bridge and time)

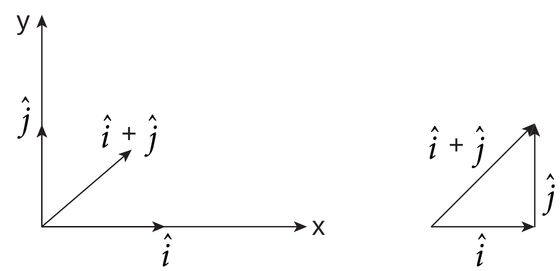
$$\text{Then, Speed} = 60\text{ km h}^{-1}$$
$$= 60 \times \frac{5}{18}\text{ m s}^{-1} = \frac{150}{9}\ ms^{-1}$$

Then, Time taken to cover this

$$\text{distance} = \frac{1100}{150/9}\text{ s} = 66\text{ s}$$

4. Draw the resultant direction of the two unit vectors $\hat{i}$ and $\hat{j}$. Use a 2-dimensional Cartesian co-ordinate system. Is $\hat{i} + \hat{j}$ a unit vector?

By using the triangular law of addition $\hat{i} + \hat{j}$ as shown in the following figure,



The definition of unit vector is $\hat{A}.\hat{A} = 1$

But here,

$$(\hat{i} + \hat{j}).(\hat{i} + \hat{j}) = \hat{i}.\hat{i} + \hat{i}.\hat{j} + \hat{j}.\hat{i} + \hat{j}.\hat{j}$$
$$= 1 + 0 + 0 + 1 = 2$$

So, $\hat{i} + \hat{j}$ is not a unit vector.

To make any vector to a unit vector, must divide the vector by its magnitude,

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

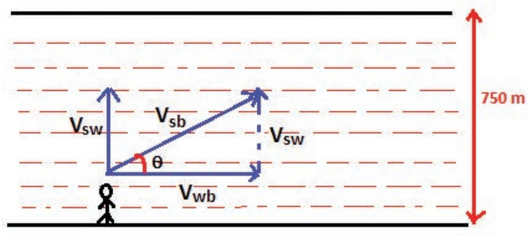
The norm of the vector $\hat{i} + \hat{j} = \sqrt{2}$.

Hence, the unit vector is $\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

5. A swimmer moves across the Cauvery river of 750 m wide. The velocity of the swimmer relative to water ($\vec{v}_{sw}$) is 1.5 m s^{-1} and directed perpendicular to the water current. The velocity of water relative to the bank ($\vec{v}_{wb}$) is 1 m s^{-1}. Calculate the
(a) velocity of the swimmer with respect to the bank of the river ($\vec{v}_{sb}$).
(b) time taken by the swimmer to cross the Cauvery river.

Solution:

(a) We can draw the following picture from the given data in the problem.



The velocity of the swimmer relative to the bank $\vec{v}_{sb} = \vec{v}_{sw} + \vec{v}_{wb}$

Since the swimmer travels in the perpendicular direction against the water current

The magnitude is given by

$$\left|\vec{v}_{sb}\right| = \sqrt{v_{sw}^2 + v_{wb}^2} =$$
$$\sqrt{1.5^2 + 1^2} = \sqrt{3.25}\, m s^{-1} \cong 1.802\, m s^{-1}$$

The direction of the swimmer relative to the bank is given by

$$tan\theta = \frac{v_{sw}}{v_{wb}} = \frac{1.5}{1} = 1.5$$
$$\theta = tan^{-1}(1.5) \approx 56°$$

(c) The time taken by the swimmer to cross the river is equal to the total distance covered by the swimmer with velocity 1.802 ms^{-1}.

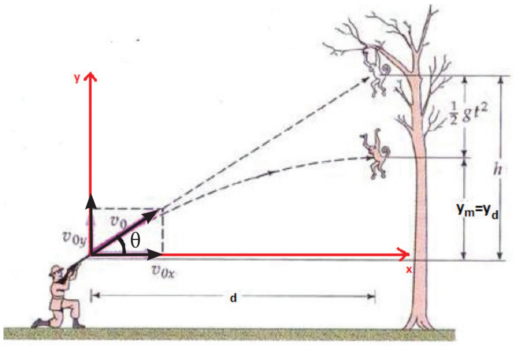
The total distance covered by him,

$$d = \frac{width\ of\ the\ river}{\sin 56^{\circ}} = \frac{750}{0.829} = 904.7\, m$$

The time taken by the swimmer,

$$T = \frac{d}{v_{sb}} = \frac{904.7}{1.802} \approx 502\, s$$

6. A monkey hangs on a tree. A hunter aims a gun at the monkey and fires the bullet with velocity v_0 which makes angle θ with horizontal direction. At the instant gun fires, monkey leaves the branch and falls straight down to escape from the bullet as shown in the figure. Will bullet hit the monkey or will the monkey escape the bullet? (ignore air resistance)



$$y_m = y_d = v_{oy}t - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2$$

As soon as the monkey begins to fall, it will have downward vertical motion with acceleration due to gravity g.

Its equation of motion at any time t is given by

$$y_m = h - \frac{1}{2}gt^2 \qquad (1)$$

When the bullet comes out of the gun, it has both vertical and horizontal components of velocity given by

$$v_{0x} = v_0 \cos\theta ; v_{0y} = v_0 \sin\theta \qquad (2)$$

Let us assume the horizontal distance between the monkey and hunter is 'd'.

At time t, the horizontal distance travelled by the bullet $x = v_0 \cos\theta\, t$.

When the horizontal position of bullet, $x = d$, the time $d = v_{0x}T$. It implies that $T = d / v_{0x}$

At this time T, the vertical distance covered by the bullet is

$$y_b = v_{0y}T - \frac{1}{2}gT^2 = \frac{v_{0y}d}{v_{0x}} - \frac{1}{2}gT^2.$$

$$y_b = \frac{d\,v_0\,sin\theta}{v_0\,cos\theta} - \frac{1}{2}gT^2$$

$$= d\,tan\theta - \frac{1}{2}gT^2 \qquad (3)$$

But from the figure we can write, $tan\theta = \frac{h}{d}$.

$$h = d\,tan\theta\,.$$

By substituting this in the equation (3), we get,

$$y_b = h - \frac{1}{2}gT^2 \qquad (4)$$

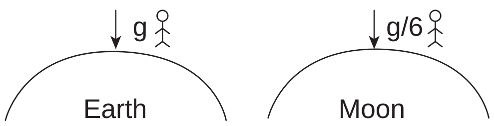
At this same time T, the vertical position of the monkey can be calculated from the equation (1)

$$y_m = h - \frac{1}{2}gT^2 \qquad (5)$$

Note that at the time T, the y coordinate of both monkey and bullet is same. It implies that the bullet will hit the monkey.

7. A three storey building of height 100 m is located on Earth and a similar building is also located on Moon. If two persons jump from the top of these buildings on Earth and Moon simultaneously, when they reach the ground, what will be their speed? (g = 10 m s^{-2})

Solution:



For both persons, the Kinematic equations are the same, with u = 0, a_e = g and $a_{moon} = \frac{g}{6}$. Then

$$a_e = g \text{ and } a_m = \frac{g}{6}$$

For a person on earth, $V_{earth} = \sqrt{2gh}$ $= \sqrt{2g100} = \sqrt{2 \times 10 \times 100}$

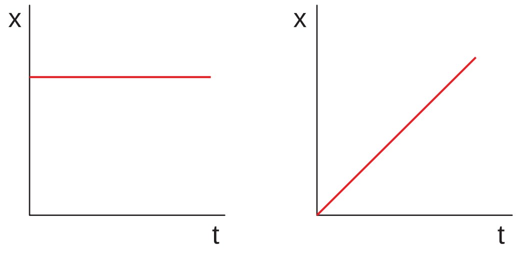
Hence, $V_{earth} = \sqrt{2000}\ m\ s^{-1}$ gives the velocity at the ground, on earth.

Similarly, for a person on the moon,

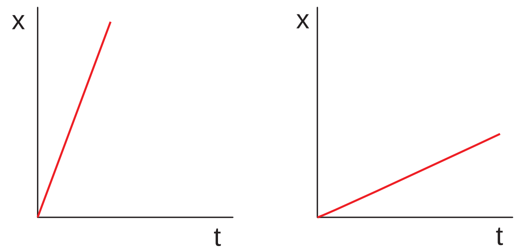
$$V_{moon} = \sqrt{\frac{2gh}{6}} = \frac{\sqrt{2000}}{\sqrt{6}}\ m\ s^{-1}$$

The person on earth reaches ground with greater velocity than the person on the moon

8. The following graphs represent position – time graphs. Arrange the graphs in ascending order of increasing speed.



(a) (b)



(c) (d)

The slope in the position – time graph will give the speed of the particle.

In the graph (a) slope is zero. Graph (c) has higher slope than graphs (b) and (d). So we can arrange the speeds in ascending order as

$$v_a < v_d < v_b < v_c$$

SOLVED EXAMPLE UNIT-3

1. A body of mass 100 kg is moving with an acceleration of 50 cm s^{-2}. Calculate the force experienced by it.

Solution:

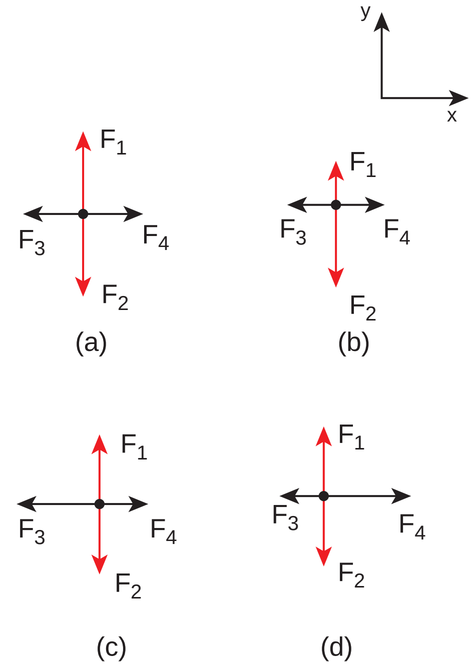
Mass m = 100 kg
Acceleration a = 50 cm s^{-2} = 0.5 m s^{-2}
Using Newton's second law,

$$F = ma$$

$$F = 100 \text{ kg x } 0.5 \text{ m s}^{-2} = 50 \text{ N}$$

2. Identify the free body diagram that represents the particle accelerating in positive *x* direction in the following.

The relative magnitude of forces should be indicated when the free body diagram for mass m is drawn.



Case (a):

The forces F_1 and F_2 have equal length but opposite direction. So net force along y-direction is zero. Since the force is zero, acceleration is also zero along Y-direction (Newton's second law). Similarly in the x direction, F_3 and F_4 have equal length and opposite in direction. So net force is zero in the x direction. So there is no acceleration in x direction.

Case (b):

The forces F_1 and F_2 are not equal in length and act opposite to each other. The figure (b) shows that there are unbalanced forces along the y-direction. So the particle has acceleration in the -y direction. The forces F_3 and F_4 are having equal length and act in opposite directions. So there is no net force along the x direction. So the particle has no acceleration in the x direction.

Case (c):

The forces F_1 and F_2 are equal in magnitude and act opposite to each other. The net force is zero in y direction. So in y-direction there is no acceleration. The forces F_3 and F_4 are not equal in magnitude and F_3 is greater than F_4. So there is a net acceleration in negative x direction

Case (d):

The forces F_1 and F_2 are equal in magnitude and act opposite to each other. The net force is zero in y direction. So there is no acceleration in y-direction. The forces F_3 and F_4 are not equal in magnitude. The force F_4 is greater than the force F_3. So there is a net acceleration in the positive x direction.

3. A gun weighing 25 kg fires a bullet weighing 30 g with the speed of 200 ms^{-1}. What is the speed of recoil of the gun.

Solution:

Mass of the gun M = 25 kg
Mass of the bullet m = 30 g = 30×10^{-3} kg
Speed of bullet v = 200 m s^{-1}
Speed of gun V = ?
The motion is in one dimension.
As per law of conservation of momentum,

$$MV + mv = 0$$

$$V = \frac{-mv}{M}$$

$$V = \frac{-30 \times 10^{-3} \times 200}{25} = -240 \times 10^{-3}\, m s^{-1}$$

The negative sign shows that the gun moves in the opposite direction of the bullet. Further the magnitude of the recoil speed is very small compared to the bullet's speed.

4. A wooden box is lying on an inclined plane. What is the coefficient of friction if the box starts sliding when the angle of inclination is 45°.

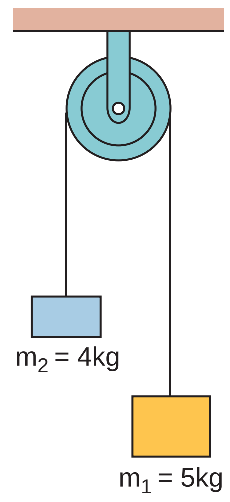
Solution:

Angle of inclination $\Theta = 45°$
$\therefore$ Coefficient of friction $\mu = \tan\Theta = \tan 45° = 1$

5. Two masses m_1 = 5 kg and m_2 = 4 kg tied to a string are hanging over a light frictionless pulley. What is the acceleration of each mass when left free to move? (g = 10 m s^{-2})

$$a = \frac{m_1 - m_2}{m_1 + m_2} \times g$$

$$= \frac{5-4}{5+4} \times 10 = \frac{1}{9} \times 10 = 1.1\, m s^{-2}$$



6. A block of mass m is pushed momentarily along a horizontal surface with an initial velocity u. If μ_k is the coefficient of kinetic friction between the object and surface, find the time at which the block comes to rest.

Solution:

When the block slides, the force acting on the block is kinetic friction which is equal to $f_k = \mu_k mg$

From Newton's second law $ma = -\mu_k mg$

The negative sign implies that force acts on the opposite direction of motion.

The acceleration of the block while sliding $a = -\mu_k g$.

The negative sign implies that the acceleration is in opposite direction of the velocity.

Note that the acceleration depends only on g and the coefficient of kinetic friction μ_k

We can apply the following kinematic equation

$$v = u + at$$

The final velocity is zero.

$$0 = u - \mu_k gt$$

$$t = \frac{u}{\mu_k g}$$

7. Three blocks of masses 10 kg, 7 kg and 2 kg are placed in contact with each other on a frictionless table. A force of 50 N is applied on the heaviest mass. What is the acceleration of the system?

Solution:

We know that

$$a = \left[\frac{F}{m_1 + m_2 + m_3}\right] = \frac{50N}{10\,kg + 7\,kg + 2\,kg}$$

$$= \frac{50}{19} = 2.63\,m\,s^{-2}$$

8. The coefficient of friction between a block and plane is $\frac{1}{\sqrt{3}}$. If the inclination of the plane gradually increases, at what angle will the object begin to slide?

Since the coefficient of friction is $\frac{1}{\sqrt{3}}$

$$\text{Tan } \Theta = \frac{1}{\sqrt{3}} \Rightarrow \Theta = 30°$$

9. Find the maximum speed at which a car can turn round a curve of 36 m radius on a level road. Given the coefficient of friction between the tyre and the road is 0.53.

Radius of the curve r = 36 m

Coefficient of friction $\mu = 0.53$

Acceleration due to gravity $g = 10\ ms^{-2}$

$$v_{max} = \sqrt{\mu rg} = \sqrt{0.53 \times 36 \times 10}$$

$$= 13.81\,m\,s^{-1}$$

10. Calculate the centripetal acceleration of the Earth which orbits around the

Sun. The Sun to Earth distance is appriximately 150 million km. (Assume the orbit of Earth to be circular)

The centripetal acceleration $a_c = \frac{v^2}{r}$

V - velocity of Earth around the orbit

r - radius of orbit or distance of Earth to Sun

Velocity of Earth is written in terms of angular velocity (ω) as

$$v = \omega r$$

By substituting in the centripetal acceleration formula, $a_c = \frac{\omega^2 r^2}{r} = \omega^2 r$

But $\omega = \frac{2\pi}{T}$ where T is time for the Earth to orbit around the sun, which is one year.

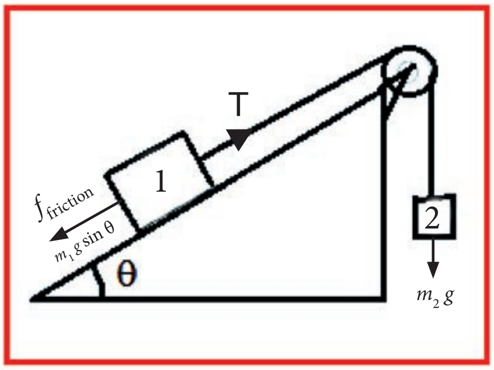
$$\mathrm{T} = 365 \text{ days} = 365 \times 24 \times 60 \times 60 \text{ s} = 3.1 \times 10^7 \, s$$

$$\omega = 2.02 \times 10^{-7} \, rad \ per \ sec$$

$$a_c = \left(2.02 \times 10^{-7}\right)^2 \times \left(150 \times 10^9\right)$$

$$a_c = 6.12 \times 10^{-3} \, m s^{-2}$$

11. A block 1 of mass m_1, constrained to move along a plane inclined at an angle θ to the horizontal, is connected via a frictionless, massless and inextensible string that passes over a massless pulley, to a second block 2 of mass m_2. Assume the coefficient of static friction between the block and the inclined plane is μ_s and the coefficient of kinetic friction is μ_k



What is the relation between the masses of block 1 and block 2 such that the system just starts to slip?

Solution:

For all parts of this problem, it will be convenient to use different coordinate systems for the two different blocks. For block 1, take the positive x -direction to be up the incline, parallel to the plane, and the positive y -direction to be perpendicular to the plane, directed with a positive upward component. Take the positive direction of the position of block 2 to be downward.

The normal component N of the contact force between block 1 and the ramp will be

$$N = m_1 g \cos\theta \, . \qquad (1)$$

The net x -component of the force on block 1 is then

$$F_{1x} = T - f_{\text{friction}} - m_1 g \sin\theta \qquad (2)$$

where T is the tension in the string

For the just-slipping condition, the frictional force has magnitude

$$f_{\text{friction}} = \mu_s N = \mu_s m_1 g \cos\theta. \qquad (3)$$

The tension in the string is the gravitational force of the suspended mass,

$$T = m_2 g. \quad (4)$$

For the just-slipping condition, the net force on block 1 must be zero. Equations (2), (3) and (4) gives

$$0 = m_2 g - \mu_s m_1 g \cos\theta - m_1 g \sin\theta$$

$$m_2 = m_1(\mu_s \cos\theta + \sin\theta)$$

12. Consider two objects of masses 5 kg and 20 kg which are initially at rest. A force 100 N is applied on the two objects for 5 second.

a) What is the momentum gained by each object after 5 s.

b) What is the speed gained by each object after 5 s.

Final momentum on each object $\Delta P = F\Delta t = 100\text{X}\ 5 = 500\ \text{kgms}^{-1}$

Final speed on the object of mass 5 kg = 500 /5 =100 m s^{-1}

Final speed on the object of mass 20 kg =500/20 =25 m s^{-1}

Note that momentum on each object is the same after 5 seconds but speed is not the same after 5 seconds. The heavier mass acquires lesser speed than the one with lower mass.

13. An object of mass 5 kg is initially at rest on the surface. The surface has coefficient of kinetic friction $\mu_k = 0.6$. What initial velocity must be given to the object so that it travels 10 m before coming to rest?

When the object moves on the surface it will experience three forces.

a) Downward gravitational force (mg)

b) Upward normal force (N)

c) Frictional force opposite to the motion of the object.

Since there is no motion along the vertical direction, magnitude of normal force is equal to the magnitude of gravitational force.

$$N = mg$$

Applying Newton's second law along the x direction

$$m\vec{a} = -\mu_k mg\hat{i}$$

The acceleration is $\vec{a} = -\mu_k mg\hat{i}$

Note that the acceleration is along the x direction since the frictional force acts along the negative x direction.

$$\text{Or} \quad a = -\mu_k g$$

Note that the acceleration is uniform during the entire motion. We can use Newton's kinematic equation to find the final velocity.

Along the x direction $v^2 = u^2 + 2as$

Here v = final velocity and u = initial velocity to be given to travel a distance s.

In this problem s = 10 m

Since the particle comes to rest, the final velocity v = 0

$$0 = u^2 - 2\mu_k gs$$

$$u = \sqrt{2\mu_k gs}$$

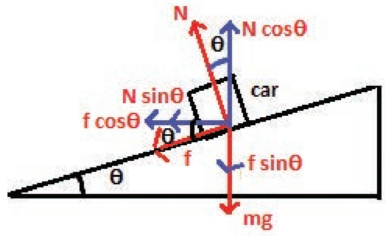
$$u = \sqrt{2 \times 0.6 \times 9.8 \times 10} = 10.8\ ms^{-1}$$

14. In the section 3.7.3 (Banking of road) we have not included the friction exerted by the road on the car. Suppose the coefficient of static friction between the car tyre and the surface of the road is μ_s, calculate the minimum speed with which the car can take safe turn?

When the car takes turn in the banked road, the following three forces act on the car.

(1) The gravitational force mg acting downwards
(2) The normal force N acting perpendicular to the surface of the road
(3) The static frictional force f acting on the car along the surface.

The following figure shows the forces acting on the horizontal and vertical direction.



When the car takes turn with the speed v, the centripetal force is exerted by horizontal component of normal force and static frictional force. It is given by

$$N\sin\theta + f\cos\theta = \frac{mv^2}{r} \qquad (1)$$

In the vertical direction, there is no acceleration. It implies that the vertical component of normal force is balanced by downward gravitational force and downward vertical component of frictional force. This can be expressed as

$$N\cos\theta = mg + f\sin\theta$$

Or $$N\cos\theta - f\sin\theta = mg \qquad (2)$$

Diving the equation (1) by equation (2), we get

$$\frac{N\sin\theta + f\cos\theta}{N\cos\theta - f\sin\theta} = \frac{v^2}{rg} \qquad (3)$$

To calculate the maximum speed for the safe turn, we can use the maximum static friction is given by $f = \mu_s N$. By substituting this relation in equation (3), we get

$$\frac{N\sin\theta + \mu_s N\cos\theta}{N\cos\theta - \mu_s N\sin\theta} = \frac{v_{\max}{}^2}{rg}$$

By simplifying this equation,

$$\frac{N\cos\theta\left\{\left(\frac{N\sin\theta}{N\cos\theta}\right) + \mu_s\right\}}{N\cos\theta\left(1 - \mu_s\frac{N\sin\theta}{N\cos\theta}\right)} = \frac{v_{\max}{}^2}{rg}$$

$$\frac{(\tan\theta + \mu_s)}{1 - \mu_s\tan\theta} = \frac{v_{\max}{}^2}{rg}$$

The Maximum speed for safe turn is given

by $$v_{\max} = \sqrt{rg\frac{(\tan\theta + \mu_s)}{(1 - \mu_s\tan\theta)}} \qquad (4)$$

Suppose we neglect the effect of friction ($\mu_s = 0$), then safe speed

$$v_{safe} = \sqrt{rg\tan\theta} \qquad (5)$$

Note that the maximum speed with which the car takes safe turn is increased by friction (equation (4)). Suppose the car turns with speed $v < v_{safe}$, then the static friction acts up in the slope to prevent from inward skidding.

If the car turns with the speed little greater than, then the static friction acts down the slope to prevent outward skidding. But if the car turns with the speed much greater than v_{safe} then static friction cannot prevent from outward skidding.

SOLVED EXAMPLE UNIT-4

1. A force $\vec{F} = \hat{i} + 2\hat{j} + 3\hat{k}$ N acts on a particle and displaces it through a distance $\vec{S} = 4\hat{i} + 6\hat{j}$ m. Calculate the work done.

Solution:

Force $\vec{F} = \hat{i} + 2\hat{j} + 3\hat{k}$ N

Distance $\vec{S} = 4\hat{i} + 6\hat{j}$ m

Work done = $\vec{F} \cdot \vec{S} = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (4\hat{i} + 6\hat{j})$

= 4+12+0 = 16 J

2. A particle moves along X- axis from x=0 to x=8 under the influence of a force given by F= $3x^2 - 4x + 5$. Find the work done in the process.

Solution:

Work done in moving a particle from x=0 to x=8 will be

$$W = \int_0^8 F dx = \int_0^8 (3x^2 - 4x + 5) dx$$

$$= \left[\frac{3x^3}{3} - \frac{4x^2}{2} + 5x \right]_0^8$$

$$W = \left[3\frac{(8)^3}{3} - 4\left(\frac{8^2}{2}\right) + 40 \right]$$

$$= [512 - 128 + 40] = 424 J$$

3. A body of mass 10kg at rest is subjected to a force of 16N. Find the kinetic energy at the end of 10 s.

Solution:

Mass m = 10 kg

Force F = 16 N

time t = 10 s

$$a = F/m = \frac{16}{10} = 1.6 m s^{-2}$$

we know that, $v = u + at$

$= 0 + 1.6 \times 10 = 16\ m\ s^{-1}$

Kinetic energy K.E $= \frac{1}{2} mv^2$

$= \frac{1}{2} \times 10 \times 16 \times 16$

$= 1280 J$

4. A body of mass 5kg is thrown up vertically with a kinetic energy of 1000 J. If acceleration due to gravity is $10\ m\ s^{-2}$, find the height at which the kinetic energy becomes half of the original value.

Solution:

Mass m = 5 kg

K.E E = 1000 J

g = $10\ m\ s^{-2}$

At a height 'h', mgh = $\frac{E}{2}$

$$5 \times 10 \times h = \frac{1000}{2}$$

$$h = \frac{500}{50} = 10 m$$

5. Two bodies of masses 60 kg and 30 kg moving in the same direction along straight line with velocity 40 cm s^{-1} and 30 cm s^{-1}

respectively suffer one dimensional elastic collision. Find their velocities after collision.

Solution:

Mass m_1 = 60 kg

Mass m_2 = 30 kg

$$V_1 = 40\ cms^{-1}$$
$$V_2 = 30\ cms^{-1}$$

Solution:

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$
$$v_2 = \frac{(m_2 - m_1)}{m_1 + m_2}u_2 + \frac{2m_1}{m_1 + m_2}u_1$$

Substituting the values, we get,

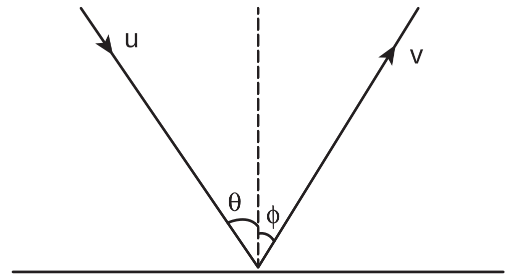
$$v_1 = \frac{(60-30)}{90}\times 40 + \frac{2\times 30}{90}\times 30$$

$$v_1 = \frac{1}{90}\ [1200+1800]$$
$$= \frac{3000}{90} = 33.3 cms^{-1}$$

Likewise,

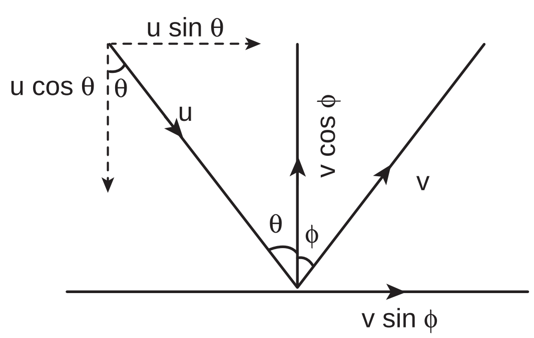
$$v_2 = \frac{(30-60)}{90}\times 30 + \frac{2\times 60}{90}\times 40$$
$$v_2 = \frac{1}{90}[-900 + 4800]$$
$$= \frac{3900}{90} = 43.3 cms^{-1}$$

6. A particle strikes a horizontal frictionless floor with a speed u at an angle θ with the vertical and rebounds with the speed v at an angle ϕ with the vertical. The coefficient of restitution between the particle and floor is e. What is the magnitude of v?



Solution:

Applying component of velocities,



The x - component of velocity is

$$u\sin\theta = v\sin\phi \qquad (1)$$

The magnitude of y – component of velocity is not same, therefore, using coefficient of restitution,

$$e = \frac{v\cos\varphi}{u\cos\theta} \qquad (2)$$

Squaring (1) and (2) and adding we get

$$v^2\sin^2\varphi = u^2\sin^2\theta$$
$$v^2\cos^2\varphi = e^2u^2\cos^2\theta$$
$$adding$$
$$v^2 = u^2\sin^2\theta + e^2u^2\cos^2\theta$$
$$\therefore v^2 = u^2\left[\sin^2\theta + e^2\cos^2\theta\right]$$
$$v = u\sqrt{\sin^2\theta + e^2\cos^2\theta}$$

7. A particle of mass m is fixed to one end of a light spring of force constant k and un-stretched length l. It is rotated with an angular velocity ω in horizontal circle. What will be the length increase in the spring?

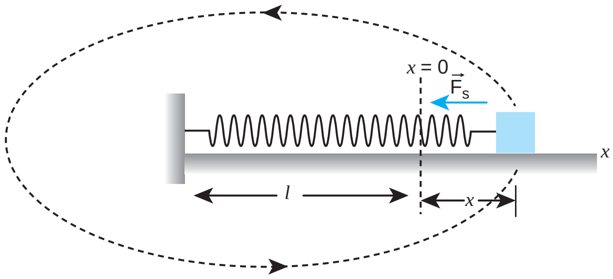
Solution:

Mass of particle = m

Force constant = k

Un-stretched length = l

Angular velocity = ω



Let 'x' be the increase in the length of the spring.

The new length = $(l+x) = r$

When the spring is rotated in a horizontal circle,

Spring force = centripetal force.

$$kx = m\omega^2(l+x)$$

$$x = \frac{m\omega^2 l}{k-m\omega^2}$$

8. A gun fires 8 bullets per second into a target X. If the mass of each bullet is 3 g and its speed 600 ms^{-1}, then calculate the power delivered by the bullets.

Solution:

Power = work done per second = total kinetic energy of 8 bullets per second

$$P = 8\times(\text{kinetic energy of each bullet per second})$$

$$= 8\times\frac{1}{2}\times\left(3\times10^{-3}\right)\times\left(600\right)^2$$

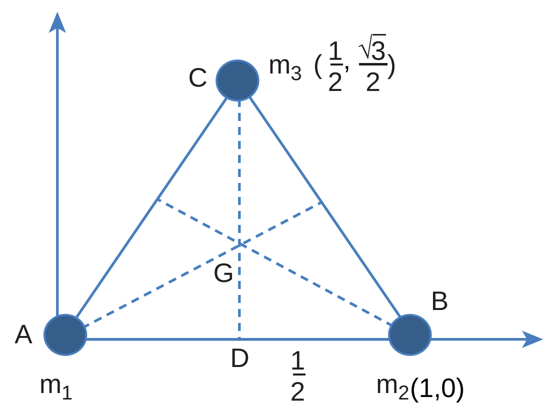
P = 4320 W

P = 4.320 kW

SOLVED EXAMPLE UNIT-5

1. Three particles of masses m_1 = 1 kg, m_2 = 2 kg and m_3 = 3 kg are placed at the corners of an equilateral triangle of side 1m as shown in Figure. Find the position of centre of mass.

Solution:



The centre of mass of an equilateral triangle lies at its geometrical centre G.

The positions of the mass m_1, m_2 and m_3 are at positions A, B and C as shown in the Figure.

From the given position of the masses, the coordinates of the masses m_1 and m_2 are easily marked as (0,0) and (1,0) respectively.

To find the position of m_3 the Pythagoras theorem is applied. As the ΔDBC is a right angle triangle,

$$BC^2 = CD^2 + DB^2$$

$$CD^2 = BC^2 - DB^2$$

$$CD^2 = 1^2 - \left(\frac{1}{2}\right)^2 = 1 - \left(\frac{1}{4}\right) = \frac{3}{4}$$

$$CD = \frac{\sqrt{3}}{2}$$

The position of mass m_3 is

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ or } \left(0.5, 0.5\sqrt{3}\right)$$

X Coordinate of centre of mass,

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$x_{CM} = \frac{(1\times 0) + (2\times 1) + (3\times 0.5)}{1+2+3} = \frac{3.5}{6}$$

$$x_{CM} = \frac{7}{12}\text{ m}$$

Y Coordinate of centre of mass,

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$y_{CM} = \frac{(1\times 0) + (2\times 0) + (3\times 0.5\times\sqrt{3})}{1+2+3} = \frac{1.5\sqrt{3}}{6}$$

$$y_{CM} = \frac{\sqrt{3}}{4}\text{ m.}$$

$\therefore$ The coordinates of centre of mass G

(x_{CM}, y_{CM}) is $\left(\frac{7}{12}, \frac{\sqrt{3}}{4}\right)$

2. An electron of mass $9 \times 10^{-31} kg$ revolves around a nucleus in a circular orbit of radius 0.53 Å. What is the angular momentum of the electron? (Velocity of electron is, $v = 2.2 \times 10^6 m s^{-1}$)

Solution:

Mass of the electron, $m = 9 \times 10^{-31} kg$

Radius of the electron, r = 0.53 Å = $0.53 \times 10^{-10} m$

Velocity of the electron, $v = 2.2 \times 10^6 m s^{-1}$

Angular momentum of electron is, $L = I\,\omega$

Electron is considered as a point mass. Hence, its moment of inertia is, $I = m r^2$

The relation, $\omega = \frac{v}{r}$ could be used.

Angular momentum, $L = mr^2 \times \frac{v}{r}$

$= mvr$

$$= 9.1 \times 10^{-31} \times 2.2 \times 10^6 \times 0.53 \times 10^{-10}$$

$$L = 1.06 \times 10^{-34}\, kg\, m^2\, s^{-1}$$

3. A solid sphere of mass 20 kg and radius 0.25 m rotates about an axis passing through the centre. What is the angular momentum if the angular velocity is 5 rad s^{-1}

Solution:

Mass of the sphere, m = 20 kg

Radius r = 0.25 m

Angular velocity $\omega = 5 \text{ rad s}^{-1}$

Solution:

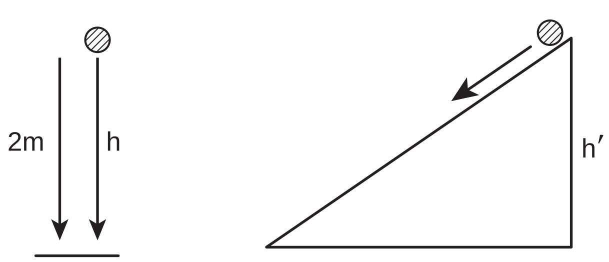
Angular momentum $L = I\omega = \frac{2}{5}mr^2\omega$

$$= \frac{2}{5} \times 20 \times (0.25)^2 \times 5 = 40 \times (0.0625) = 2.5$$

$$L = 2.5 \text{ kg m}^2\text{s}^{-1}$$

4. A solid cylinder when dropped from a height of 2 m acquires a velocity while reaching the ground. If the same cylinder is rolled down from the top of an inclined plane to reach the ground with same velocity, what must be the height of the inclined plane? Also compute the velocity.

Solution:



In the first case,

potential energy = kinetic energy

$$\text{mgh} = \frac{1}{2}mv^2$$

$$\text{mg}\times 2 = \frac{1}{2}mv^2 \qquad (1)$$

In second case,

potential energy = translational kinetic energy + rotational kinetic energy

$$\text{mgh}' = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$\text{mgh}' = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{v^2}{r^2}\right)$$

$$\therefore \text{mgh}' = \frac{3}{4}mv^2 \qquad (2)$$

Dividing (2) by (1),

$$\frac{mgh'}{mg \times 2} = \frac{\frac{3}{4}mv^2}{\frac{1}{2}mv^2} = \frac{3}{4} \times \frac{2}{1} = \frac{3}{2}$$

$$h' = 3\,m$$

From equation (1), $2mg = \frac{1}{2}mv^2$

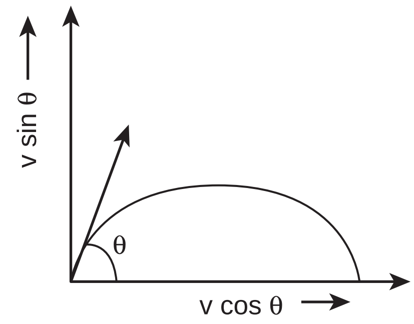
$$v = \sqrt{4g} = 2\sqrt{g}$$

$$v = 2 \times \sqrt{9.81}$$

$$v = 6.3\,m\,s^{-1}$$

5. A small particle of mass m is projected with an initial velocity v at an angle θ with x axis in X-Y plane as shown in Figure. Find the angular momentum of the particle.

Solution:



Let the particle of mass m cross a horizontal distance x in time t.

$$\text{Angular momentum } \vec{L} = \int \vec{\tau}\, dt$$

$$\text{But } \vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{r} = x\hat{i} + y\hat{j} \text{ and } \vec{F} = -mg\hat{j}$$

$$\therefore \vec{\tau} = \left(x\hat{i} + y\hat{j}\right) \times \left(-mg\hat{j}\right)$$

$$\vec{\tau} = -mgx\left(\hat{i} \times \hat{j}\right) = -mgx\hat{k}$$

$$\vec{L} = -mg\int (xdt)\hat{k} = -mgv\cos\theta\left(\int tdt\right)\hat{k}$$

Let initial time t = 0 and final time $t = t_f$

$$\therefore \vec{L} = -mgv\cos\theta\left(\int_0^{t_f} tdt\right)\hat{k} = -\frac{1}{2}mgv\cos\theta t_f^2\ \hat{k}$$

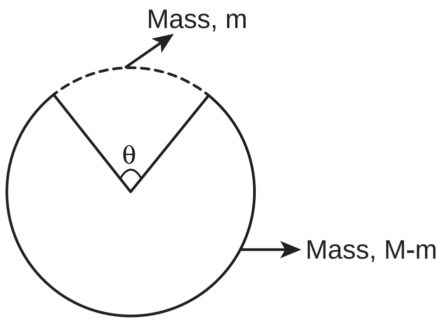
Negative sign indicates, $\vec{L}$ point inwards

6. From a complete ring of mass M and radius R, a sector angle θ is removed. What is the moment of inertia of the incomplete ring about axis passing through the centre of the ring and perpendicular to the plane of the ring?

Solution:

Let R be the radius of the ring and M be the total mass of the complete ring.

Let m be the mass of the section removed from the ring then, mass of the incomplete ring is M-m



Let us introduce a positive integer (n), such that, $n\theta = 360^\circ$, or $n = \dfrac{360^\circ}{\theta}$

$$mass\ of\ incomplete\ ring = M - m$$

$$m = \frac{M}{360} \times \theta$$

$$\therefore mass\ of\ incomplete\ ring = M - \frac{M}{360} \times \theta$$

$$mass\ of\ incomplete\ ring = M - \frac{M}{n} = M\frac{(n-1)}{n}$$

For example, $a)\ when\ \theta = 60^\circ;\ n = \dfrac{360^\circ}{60^\circ} = 6$

$$\therefore n - 1 = 5$$

$$\text{mass of incomplete ring} = \frac{5}{6}M$$

$$b)\ when\ \theta = 30^\circ,\ n = \frac{360^\circ}{30^\circ} = 12$$

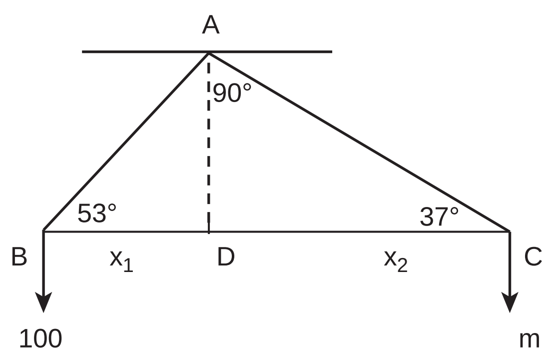
$$n - 1 = 11$$

$$\text{mass of incomplete ring} = \frac{11}{12}M$$

The moment of inertia of the incomplete ring is, $I = M\frac{(n-1)}{n}R^2$

7. A massless right angled triangle is suspended with its right angle corner. A mass of 100 kg is suspended from another corner B which subtends an angle 53°. Find the mass m that should be suspended from other corner C so that BC (hypotenuse) remains horizontal.

Solution:



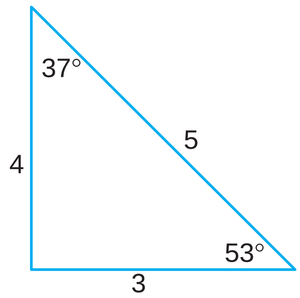
From the principle of moments,

$$100 \times g \times x_1 = m \times g \times x_2$$

$$100 \times \cos 53^\circ = m \times \cos 37^\circ \qquad (1)$$

Where, x_1 and x_2 are the arm lengths.

The right angle triangle with angles 37°, 53° and 90° is a special triangle which has the respective sides in the ratio, 3:4:5 as shown in the diagram.



Substituting the values in equation (1),

$$100 \times \cos 53^\circ = m \times \cos 37^\circ$$

$$100 \times \frac{3}{5} = m \times \frac{4}{5}$$

$$m = 100 \times \frac{3}{4}$$

$$m = 75\,kg$$

8. If energy of 1000 J is spent in increasing the speed of a flywheel from 30 rpm to 720 rpm, find the moment of inertia of the wheel.

Solution:

$$\omega_1 = 30rpm = 2\pi \times \frac{30}{60} rads^{-1} = \pi \; rads$$

$$\omega_2 = 720rpm$$

$$= 2\;\pi \times \frac{720}{60} rads^{-1} = 24\pi \; rads$$

Change in kinetic energy,

$$\Delta KE = \frac{1}{2} I(\omega_2^2 - \omega_1^2)$$

$$I = \frac{2 \times \Delta KE}{(\omega_2^2 - \omega_1^2)} = \frac{2 \times 1000}{(24\pi)^2 - (\pi)^2}$$

$$I = \frac{2000}{25\pi \times 23\pi}$$

Remember:

$a^2 - b^2 = (a + b)(a - b)$

$I \approx 0.35 \text{ kg m}^2$ as $\pi^2 \approx 10$

9. Consider two cylinders with same radius and same mass. Let one of the cylinders be solid and another one be hollow. When subjected to same torque, which one among them gets more angular acceleration than the other?

Solution:

Moment of inertia of a solid cylinder about its axis $I_s = \frac{1}{2}MR^2$

Moment of inertia of a hollow cylinder about its axis $I_h = MR^2$

$$I_s = \frac{1}{2}I_h \text{ or } I_h = 2I_s$$

$$torque\ \tau = I\alpha$$

$$\alpha = \frac{\tau}{I}$$

$$\alpha_s = \frac{\tau}{I_s} \text{ and } \alpha_h = \frac{\tau}{I_h}$$

$$\alpha_s I_s = \alpha_h I_h \Rightarrow \alpha_s = \alpha_h \frac{I_h}{I_s}$$

Since, $I_h > I_s \Rightarrow \frac{I_h}{I_s} > 1$

$$\therefore \alpha_s > \alpha_h$$

For the same torque, a solid cylinder gets more acceleration than a hollow cylinder.

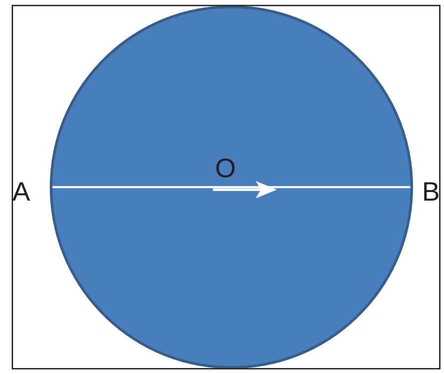
Note: The above two cylinders must be made up of materials of different density. (Say why?)

10. **A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An insect goes from A to point B along its diameter as shown in Figure. Discuss how the angular speed of the circular disc changes?**

Solution:

As the disc is freely rotating, with the insect on it, the angular momentum of the system is conserved.

$$L = I\omega = \text{constant}$$



When the insect moves towards the centre (from A to O), the moment of inertia (I) decreases. Thus, the angular velocity (ω) increases. When it moves away from centre (from O to B), the moment of inertia (I) increases. Thus, the angular velocity (ω) decreases.

11. (i) What is the shape of the graph between $\sqrt{E_{kr}}$ and L? (E_{kr} is the rotational kinetic energy and L is angular momentum)
(ii) What information can you get from the slope of the graph?
(iii) You are given the graph of $\sqrt{E_{kr}}$ and L for two bodies A and B. Which one has more moment of inertia?

Solution:

i) We know that, Rotational kinetic Energy

$$E_{kr} = \frac{1}{2}I\omega^2$$

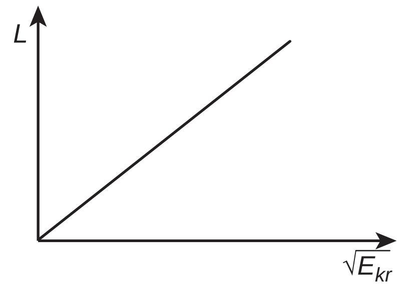
$$= \frac{1}{2}I\omega \times \omega = \frac{1}{2}.L.\omega = \frac{1}{2}\frac{L^2}{I} \qquad \because L = I\omega, \quad \omega = L/I$$

$$E_{kr} = \frac{L^2}{2I}$$

$$L^2 = 2IE_{kr}$$

$$L = \sqrt{2IE_{kr}} = \sqrt{2I}.\sqrt{E_{kr}}$$

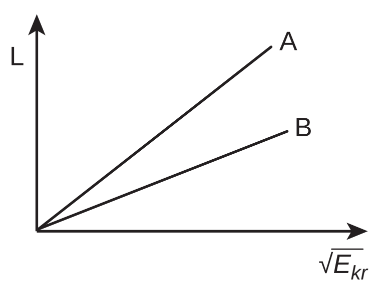
Graph of $\sqrt{E_{kr}}$ and L:



The shape of the graph is a straight line

ii) The slope of the graph gives the value of moment of inertia I.

iii) We know that the slope gives the value of moment of Inertia. The line A has higher slope and hence more moment of Inertia.



12. **Consider a thin uniform circular ring rolling down in an inclined plane without slipping. Compute the linear acceleration along the inclined plane if the angle of inclination is 45˚.**

Solution:

The linear acceleration along the inclined plane can be computed by

$$a = \frac{g\sin\theta}{1+\dfrac{K^2}{R^2}}$$

For a thin uniform circular ring, axis passing through its centre is $I = MR^2$

$$\therefore K^2 = R^2 \Rightarrow \frac{K^2}{R^2} = 1.$$

And the angle of inclination, $\theta = 45^\circ$

$\Rightarrow (\sin 45^\circ = \dfrac{1}{\sqrt{2}})$

Hence,

$$a = \frac{\dfrac{g}{\sqrt{2}}}{1+1}$$

$$a = \frac{g}{2\sqrt{2}}\,ms^{-2}$$

COMPETITIVE EXAM CORNER

MLZYF

APPENDIX 2

AWET7

Table A1.1 Systematic developments in Physics over centuries

Observation of motion of Sun, Stars, Planets and Moon	Around 3000 BC (Early Greeks)
In the universe everything is changing. Nothing remains in the same state indefinitely. The idea of 'time' started with this understanding.	Around 500 BC (Heraclitus).
Everything is composed of entirely small indivisible elements called 'atoms'. Development of theory of 'atomism'. But that time it is a hypothesis. No experimental proof.	End of 500 BC (Democritus and Leucippus)
■ Idea of natural laws behind every day phenomena ■ Idea of motion with gravity ■ Theory of four elements(Everything is made up of Earth, water, fire, air able to inter-transform into each other) ■ Earth is centre of universe(Hypothesis) ■ Force is required to move the object ■ Heavier object falls faster towards Earth than lighter object	Around 3rd century BC (Aristotle)
■ Earth is sphere ■ Measurement of radius of the Earth almost accurately	Around 240 BC (Eratosthenes)

Table A1.1 Systematic developments in physics over centuries (*Cont*)

▪ Sun is centre of the solar system (Hypothesis. Could not be proven by experimental evidence) ▪ Idea of Earth rotation around its own axis.	Around 2nd century BC (Aristarchus of Samos) Around 2nd century BC (Seleucia)
▪ Foundation of hydrostatics ▪ Idea of lever ▪ Mechanical work using pulleys ▪ Law of Buoyancy known as a Archimedes principle ▪ First accurate value of the number 'pi'	Around 3rd century BC (Archimedes)
▪ Focused on motion of planets and stars ▪ Prediction of solar eclipses ▪ Calculation of distance of Earth to Moon, Earth to Sun ▪ Astronomical observation were recorded	End of 2nd century BC (Hippachrus)
▪ Geo centric model (Not hypothesis. Explained a lot of naked eye observation) ▪ Explanation of Planets 'retrograde motion' ▪ "Almagest"- First book on astronomy	Around 100 AD(CE) (Ptolemy)
▪ Idea of Earth's rotation about its own axis ▪ Idea of zero ▪ Contribution to mathematics	5th Century AD (Aryabhatta-India)
▪ Understanding of early optics ▪ Boon on 'Treasury of astronomy'- Accurate astronomical table than Ptolemy's data	9th century AD (Ibn al-Hayatham- Arabia) 12th Century AD (Nasir al-Din- Persian astronomer)
▪ From 7th century to 14th century major development in science happened in muslim countries (Arabia, Persia, Iran etc.)	

Table A1.1 Systematic developments in physics over centuries (*Cont*)

▪ Copernicun Revolution ▪ Heliocentric model (Not hypothesis. It provides simplest explanation than Ptolemy model for motion of stars and Planets) ▪ Accurate astronomical datas ▪ Laws of Planetary motion	15th Century AD AD 1543 (Copernicus) Tycho Brahe Kepler
▪ Law of inertia ▪ Telescope observation(Founder of modern observational astronomy) ▪ Calculation of time period of the Moons of the Jupiter ▪ Earth is not flat ▪ All object fall to Earth at the same rate (Disproved Aristotle's argument) ▪ Law of inertia(Force need not required to maintain the motion- disproved Aristotle's argument) ▪ Pendulum, inclined plane experiments ▪ Study of projectile motion ▪ Introduction of 'controlled experiments'	Gallieo Galilei (1564- 1642)
▪ Introduction of Cartesian coordinate system ▪ Idea of analytical geometry	Rane Descarte (1596-1650)
▪ 17th and 18th century Development	
▪ Laws of motion ▪ Quantitative idea of motion ▪ Law of gravitation ▪ Development of calculus independent of Leibnitz ▪ Founder of modern optics (reflection, dispersion, prism) ▪ Light consists of minute particles 'corpuscules' ▪ Derivation of Kepler laws ▪ Greatest book 'The principia mathematica'(1687) ▪ Wave theory of light	Isaac Newton (1642-1727) Christian Huygens; Father of modern optics

Table A1.1 Systematic developments in physics over centuries (*Cont*)

▪ Work on magnetism	William Gilbert (Around 1600)
▪ Behavior of gases	Robert Boyle (1627- 1691)& Robert Hooke
▪ Fluid dynamics - Bernoulli's theorem (1734)	Daniel Bernoulli(1700- 1782)
▪ Early ideas of kinetic theory of gases	Daniel Bernoulli (1700 - 1782)
▪ Study of spring motion(Hooke's law)	Robert Hooke
▪ Derivation of frequency of vibration in strings(1714)	Taylor
▪ Reformulation of Newtonian mechanics using Energy approach (Lagrangian mechanics)	De Alembert, Lagrange
▪ Invention of steam engine(1781)-	James Watt
▪ Force between charges (Coulomb's law)	Coulomb
▪ 19^{th} century development	
▪ Early ideas of Thermodynamics(1840s)	James Joule, Carnot
▪ Caloric theory	
▪ Laws of thermodynamics(1850s)	Kelvin, Clausius
▪ Behavior of gases, velocity and speed(1860s)	James Clark Maxwell
▪ Foundation of statistical mechanics and entropy formula (around 1870s)	Boltzmann
▪ Wave nature of light- Experiments	Young and Fresnel
▪ Behavior of electric charges	
▪ Magnetic effect of electric current(1820s)	Oersted
▪ Force between two parallel currents	Ampere
▪ Principle of Least action, Hamilton mechanics (1821)	William Hamilton
▪ Electric powered motor, electricity demonstration	Micheal Faraday
▪ Theory of electromagnetism(1873)- Bridge between electricity and magnetism	James Clerk Maxwell
▪ Maxwell equations –Paved way to modern technology	
▪ Alternating current	Tesla

Table A1.1 Systematic developments in physics over centuries (*Cont*)

▪ 20^{th} century developments	
▪ Study of black body radiation ▪ Discovery of an electron ▪ Rutherford atomic model ▪ Study on radioactivity	Max planck(around 1900) J.J.Thomson Rutherford(1910s) Marie Curie(1920s and 30s)
▪ Special theory of relativity, Photo electric effect, Existence of atom(1905), $E=mc^2$ ▪ Revolution of physics after Newton ▪ New Idea of space and time ▪ The General theory of relativity(Greatest theory of 20^{th} century)- 1915 ▪ Study of specific heat capacities	Albert Einstein
▪ Study of atoms ▪ Bohr atom model (1912) ▪ Behavior of electron, proton – Schrodinger equation ▪ Uncertainty principle ▪ Formulation of Quantum Mechanics ▪ Formulation of quantum field theory ▪ Particle physics, standard model ▪ X-ray diffraction(1930s)- Paved way to understanding the materials ▪ Raman effect ▪ Study of stars and black holes ▪ Invention of transistor(1947) ▪ Classification of stars using temperature (Astro thermodynamics), Saha ionization formula ▪ Field of Cosmology (1920s) ▪ Expanding universe model(1922) ▪ Discovery of redshift	 Niels Bohr Schrodinger Heisenberg Paul Dirac Dirac, Feynman, Schwinger Gellman, Weinberg, Abdus salam C.V. Raman(India) Chandrasekhar(Indian origin) John Bardeen, Walter Brattain,William Schokley Megnad Saha(India) Eddington, Schwarchild Thomas Friedmann Edwin Hubble

Table A1.1 Systematic developments in physics over centuries (*Cont*)

▪ Birth of materials science ▪ Nanotechnology, Condensed matter physics ▪ Gravitational waves, Dark energy, Dark matter, String theory	

APPENDIX A1.2

vi) Error involving the division of two quantities (using the method of differentiation)

Consider $Z = \frac{A^n}{B^m}$

Taking logarithms of both the sides,

$$\log Z = \log A^n - \log B^m$$
$$\log Z = n\log A - m\log B$$

Differentiating both sides, we get,

$$\frac{dZ}{Z} = n\frac{dA}{A} - m\frac{dB}{B}$$

In terms of fractional error, this may be written as,

$$\pm\frac{\Delta Z}{Z} = \pm n\frac{\Delta A}{A} \mp m\frac{\Delta B}{B}$$

Maximum possible relative error in z is given by

$$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A} + m\frac{\Delta B}{B}$$

vii) Error involving the product of three quantities (using calculus method)

let $u = x^m y^n Z^p$

Taking logarithms on both sides, we get

$$\log u = \log x^m + \log y^n + \log Z^p$$
$$\log u = m\log x + n\log y + p\log Z$$

Differentiating both sides, we get

$$\frac{du}{u} = m\frac{dx}{x} + n\frac{dy}{y} + p\frac{dz}{z}$$

In terms of fractional error, the equation maybe rewritten as,

$$\pm\frac{\Delta u}{u} = \pm m\frac{\Delta x}{x} \pm n\frac{\Delta y}{y} \pm p\frac{\Delta z}{z}$$

APPENDIX A2.1

Parallelogram Law of Vector addition:

If two non-zero vectors $\vec{A}$ and $\vec{B}$ are represented by the two adjacent sides of a parallelogram then the resultant is given by the diagonal of the parallelogram passing through the point of intersection of the two vectors.

Consider the figure 2.19 below. Here the two vectors $\vec{A}$ and $\vec{B}$ are connected by a common tail at an angle θ. The parallelogram OACB is next constructed.

Then the diagonal OC is the resultant ($\vec{R}$) passing through the common tail O.

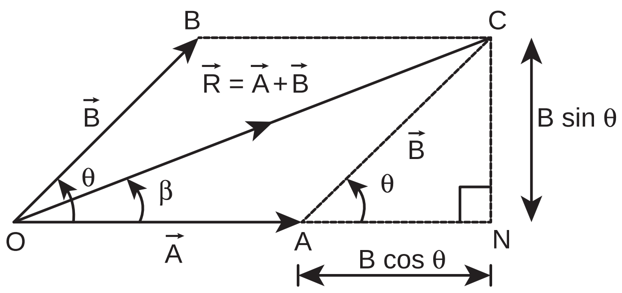


Figure 2.19 Magnitude of resultant vector by parallelogram method

We next find the magnitude and direction of this resultant vector.

(i) Magnitude:

First extend OA to the point N, so that we get ON. Then CN is drawn perpendicular to this ON, from C. Then ONC is a right angled triangle.

We can write $R^2 = ON^2 + CN^2$

$$R^2 = (OA + AN)^2 + CN^2$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\therefore R = \left|\vec{R}\right| = \left|\vec{A} + \vec{B}\right| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Special cases:

When $\theta = 0°$, then R = A+B

When $\theta = 180°$, then R = A-B

When $\theta = 90°$, then $R = \sqrt{A^2 + B^2}$

(ii) Direction

Let β be the angle between the vectors $\vec{A}$ and $\vec{R}$. Then

$$\tan\beta = \frac{CN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}$$

APPENDIX 3

Some important formulae for Integral

(1) $\int dx = x\,;\ \frac{d}{dx}(x) = 1$

(2) $\int x^n dx = \left(\frac{x^{n+1}}{n+1}\right);\ \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$

(3) $\int cudx = c\int udx$ where c is a constant

(4) $\int (u \pm v \pm w)dx = \int udx \pm \int vdx \pm \int wdx$

(5) $\int \frac{1}{x}dx = \ln x + \mathrm{C}$

(6) $\int e^x dx = e^x + \mathrm{C}$

(7) $\int \cos\theta d\theta = \sin\theta + \mathrm{C}$

(8) $\int \sin\theta d\theta = -\cos\theta\ + \mathrm{C}$

Some important formulae in Differential calculus

1. $\frac{d}{dx}(c) = 0,$ if c is a constant

2. If y = cu, where c is a constant

   and u is a function of x then
   $$\frac{dy}{dx} = \frac{d}{dx}(cu) = c\frac{du}{dx}$$

3. If y = u ±v ±w where u,v and w are functions of x then
   $$\frac{dy}{dx} = \frac{d}{dx}(u \pm v \pm w) = \frac{du}{ds} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$$

4. If $y = x^n$, where n is the real number then
   $$\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$$

5. If y = uv, where u and v are functions of x then
   $$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

6. If y is a function of x, then $dy = \frac{dy}{dx}.dx$

7. $\frac{d}{dx}(e^x) = e^x$

8. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

9. $\frac{d}{d\theta}(\sin\theta) = \cos\theta$

10. $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$

11. If y is a trigonometric function of θ and θ is the function of t, then
    $$\frac{d}{dt}(\sin\theta) = \cos\theta\frac{d\theta}{dt}$$
    $$\frac{d}{dt}(\cos\theta) = -\sin\theta\frac{d\theta}{dt}$$

APPENDIX 4

(THE GREEK ALPHABET)
(கிரேக்க எழுத்துகள்)

The Greek Alphabet	Upper Case	Lower Case
Alpha	A	α
Beta	B	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	E	ε
Zeta	Z	ζ
Eta	H	η
Theta	Θ	θ
Iota	I	ι
Kappa	K	κ
Lambda	Λ	λ
Mu	M	μ
Nu	N	ν
Xi	€	ξ
Omicron	O	o
Pi	∏	π
Rho	P	ρ
Sigma	∑	σ
Tau	T	τ
Upsilon	Y	υ
Phi	Φ	ϕ
Chi	X	χ
Psi	Ψ	ψ
Omega	Ω	ω

SOME IMPORTANT CONSTANTS IN PHYSICS

Name	Symbols	Value
Speed of light in vacuum	c	2.9979×10^{8} m s^{-1}
Gravitational constant	G	6.67×10^{-11} N m^{2} kg^{-2}
Acceleration due to gravity (sea level, at 45° latitude)	g	9.8 m s^{-2}
Planck constant	h	6.626×10^{-34} J s
Boltzmann constant	k	1.38×10^{-23} J K^{-1}
Avogadro number	N_A	6.023×10^{23} mol^{-1}
Universal gas constant	R	8.31 J mol^{-1} K^{-1}
Stefan – Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Wien's constant	b	2.898×10^{-3} m K
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Standard atmospheric pressure	1 atm	1.013×10^{5} Pa

GLOSSARY
கலைச்சொற்கள்

1.	Acceleration	-	முடுக்கம்
2.	Angular Momentum	-	கோண உந்தம்
3.	Astro physics	-	வானியற்பியல்
4.	Average Velocity	-	சராசரித் திசைவேகம்
5.	Angular displacement	-	கோண இடப்பெயர்ச்சி
6.	Angular velocity	-	கோணத் திசைவேகம்
7.	Angular acceleration	-	கோண முடுக்கம்
8.	Angle of friction	-	உராய்வுக் கோணம்
9.	Angle of repose	-	சறுக்குக் கோணம்
10.	Centripetal acceleration	-	மையநோக்கு முடுக்கம்
11.	Circular Motion	-	வட்ட இயக்கம்
12.	Concurrent forces	-	மைய விசைகள்
13.	Coplanar force	-	ஒரு தள விசைகள்
14.	Centripetal force	-	மையநோக்கு விசை
15.	Centrifugal force	-	மையவிலக்கு விசை
16.	Coefficient of friction	-	உராய்வுக் குணகம்
17.	Centre of mass	-	நிறை மையம்
18.	Couple	-	இரட்டை
19.	Centre of gravity	-	ஈர்ப்பு மையம்
20.	Circular Motion	-	வட்ட இயக்கம்
21.	Collision	-	மோதல்
22.	Conservative force	-	ஆற்றல் மாற்றா விசை
23.	Dimensional Analysis	-	பரிமாண பகுப்பாய்வு
24.	Displacement	-	இடப்பெயர்ச்சி
25.	Escape -Speed	-	விடுபடு வேகம்
26.	Elastic potential energy	-	மீட்சியழுத்த ஆற்றல்
27.	Elastic collision	-	மீட்சி மோதல்
28.	Free body diagram	-	தனித்த பொருளின் விசைப்படம்
29.	Gravitational Constant	-	ஈர்ப்பியல் மாறிலி
30.	Gravitational field	-	ஈர்ப்புப்புலம்
31.	Gravitational potential	-	ஈர்ப்பழுத்தம்
32.	Gravitational Potential energy	-	ஈர்ப்பழுத்த ஆற்றல்
33.	Gross Error	-	மொத்த பிழை
34.	Geo stationary satellite	-	புவி நிலைத் துணைக்கோள்/ செயற்கைக்கோள்
35.	Horizontal plane	-	கிடைத்தளம்
36.	Horse power	-	குதிரைத்திறன்

37. Instantaneous Velocity - உடனடி / கணத் திசைவேகம்
38. Inclined plane - சாய்தளம்
39. Inertial frame - நிலைமக்குறிப்பாயம்
40. Impulse - கணத்தாக்கு
41. Instantaneous power - உடனடித்திறன்
42. Kinetic friction - இயக்க உராய்வு
43. Linear motion - நேர்கோட்டு இயக்கம்
44. Linear density - நீள் அடர்த்தி
45. Linear momentum - நேர்கோட்டு உந்தம்
46. Least Count error - மீச்சிற்றளவு பிழை
47. Motion - இயக்கம்
48. Moment of inertia - நிலைமத்திருப்புத் திறன்
49. Non conservative force - ஆற்றல் மாற்றும் விசை
50. Orbital speed - சுற்றியக்க வேகம்
51. One dimensional motion - ஒரு பரிமாண இயக்கம்
52. accuracy - துல்லியத்தன்மை
53. Parallax Method - இடமாறு தோற்றமுறை
54. Projectile - எறியம்
55. Point mass - புள்ளி நிறை
56. Precison - நுட்பத்தன்மை
57. Planetary Motion - கோள்களின் இயக்கம்
58. Polar satellite - துருவ மைய துணைக்கோள்/ செயற்கைக்கோள்
59. Precession - அச்சுச் சுழற்சி
60. Pulley - கப்பி
61. Rolling - உருளுதல்
62. Random error - சமவாய்ப்பு பிழை
63. Rounding off - முழுமைபடுத்துதல்
64. Rest - ஓய்வு
65. Retardation - எதிர் முடுக்கம்
66. Relative velocity - சார்புத்திசைவேகம்
67. Range - கிடைத்தள தூரம்
68. Rolling friction - உருள்தலில் உராய்வு
69. Rigid body - திண்மப்பொருள்
70. Radius of gyration - சுழற்சி ஆரம்
71. Rotational Motion - சுழற்சி இயக்கம்
72. Spring Constant - சுருள் மாறிலி
73. Sliping - நழுவுதல்
74. Sliding - சறுக்குதல்
75. Systematic error - முறையான பிழை
76. Static friction - நிலை உராய்வு
77. Significant Number - முக்கிய எண்ணுரு
78. Time of flight - பறக்கும் நேரம்
79. Torque - திருப்பு விசை
80. Translational Motion - இடம்பெயர்வு இயக்கம்
81. Tension - இழுவிசை
82. Velocity - திசைவேகம்

LOGARITHM TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0.000	0.004	0.009	0.013	0.017	0.021	0.025	0.029	0.033	0.037	4	8	11	17	21	25	29	33	37
11	0.041	0.045	0.049	0.053	0.057	0.061	0.064	0.068	0.072	0.076	4	8	11	15	19	23	26	30	34
12	0.079	0.083	0.086	0.090	0.093	0.097	0.100	0.104	0.107	0.111	3	7	10	14	17	21	24	28	31
13	0.114	0.117	0.121	0.124	0.127	0.130	0.134	0.137	0.140	0.143	3	6	10	13	16	19	23	26	29
14	0.146	0.149	0.152	0.155	0.158	0.161	0.164	0.167	0.170	0.173	3	6	9	12	15	18	21	24	27
15	0.176	0.179	0.182	0.185	0.188	0.190	0.193	0.196	0.199	0.201	3	6	8	11	14	17	20	22	25
16	0.204	0.207	0.210	0.212	0.215	0.217	0.220	0.223	0.225	0.228	3	5	8	11	13	16	18	21	24
17	0.230	0.233	0.236	0.238	0.241	0.243	0.246	0.248	0.250	0.253	2	5	7	10	12	15	17	20	22
18	0.255	0.258	0.260	0.262	0.265	0.267	0.270	0.272	0.274	0.276	2	5	7	9	12	14	16	19	21
19	0.279	0.281	0.283	0.286	0.288	0.290	0.292	0.294	0.297	0.299	2	4	7	9	11	13	16	18	20
20	0.301	0.303	0.305	0.307	0.310	0.312	0.314	0.316	0.318	0.320	2	4	6	8	11	13	15	17	19
21	0.322	0.324	0.326	0.328	0.330	0.332	0.334	0.336	0.338	0.340	2	4	6	8	10	12	14	16	18
22	0.342	0.344	0.346	0.348	0.350	0.352	0.354	0.356	0.358	0.360	2	4	6	8	10	12	14	15	17
23	0.362	0.364	0.365	0.367	0.369	0.371	0.373	0.375	0.377	0.378	2	4	6	7	9	11	13	15	17
24	0.380	0.382	0.384	0.386	0.387	0.389	0.391	0.393	0.394	0.396	2	4	5	7	9	11	12	14	16
25	0.398	0.400	0.401	0.403	0.405	0.407	0.408	0.410	0.412	0.413	2	3	5	7	9	10	12	14	15
26	0.415	0.417	0.418	0.420	0.422	0.423	0.425	0.427	0.428	0.430	2	3	5	7	8	10	11	13	15
27	0.431	0.433	0.435	0.436	0.438	0.439	0.441	0.442	0.444	0.446	2	3	5	6	8	9	11	13	14
28	0.447	0.449	0.450	0.452	0.453	0.455	0.456	0.458	0.459	0.461	2	3	5	6	8	9	11	12	14
29	0.462	0.464	0.465	0.467	0.468	0.470	0.471	0.473	0.474	0.476	1	3	4	6	7	9	10	12	13
30	0.477	0.479	0.480	0.481	0.483	0.484	0.486	0.487	0.489	0.490	1	3	4	6	7	9	10	11	13
31	0.491	0.493	0.494	0.496	0.497	0.498	0.500	0.501	0.502	0.504	1	3	4	6	7	8	10	11	12
32	0.505	0.507	0.508	0.509	0.511	0.512	0.513	0.515	0.516	0.517	1	3	4	5	7	8	9	11	12
33	0.519	0.520	0.521	0.522	0.524	0.525	0.526	0.528	0.529	0.530	1	3	4	5	6	8	9	10	12
34	0.531	0.533	0.534	0.535	0.537	0.538	0.539	0.540	0.542	0.543	1	3	4	5	6	8	9	10	11
35	0.544	0.545	0.547	0.548	0.549	0.550	0.551	0.553	0.554	0.555	1	2	4	5	6	7	9	10	11
36	0.556	0.558	0.559	0.560	0.561	0.562	0.563	0.565	0.566	0.567	1	2	3	5	6	7	8	10	11
37	0.568	0.569	0.571	0.572	0.573	0.574	0.575	0.576	0.577	0.579	1	2	3	5	6	7	8	9	10
38	0.580	0.581	0.582	0.583	0.584	0.585	0.587	0.588	0.589	0.590	1	2	3	5	6	7	8	9	10
39	0.591	0.592	0.593	0.594	0.595	0.597	0.598	0.599	0.600	0.601	1	2	3	4	5	7	8	9	10
40	0.602	0.603	0.604	0.605	0.606	0.607	0.609	0.610	0.611	0.612	1	2	3	4	5	6	8	9	10
41	0.613	0.614	0.615	0.616	0.617	0.618	0.619	0.620	0.621	0.622	1	2	3	4	5	6	7	8	9
42	0.623	0.624	0.625	0.626	0.627	0.628	0.629	0.630	0.631	0.632	1	2	3	4	5	6	7	8	9
43	0.633	0.634	0.635	0.636	0.637	0.638	0.639	0.640	0.641	0.642	1	2	3	4	5	6	7	8	9
44	0.643	0.644	0.645	0.646	0.647	0.648	0.649	0.650	0.651	0.652	1	2	3	4	5	6	7	8	9
45	0.653	0.654	0.655	0.656	0.657	0.658	0.659	0.660	0.661	0.662	1	2	3	4	5	6	7	8	9
46	0.663	0.664	0.665	0.666	0.667	0.667	0.668	0.669	0.670	0.671	1	2	3	4	5	6	7	7	8
47	0.672	0.673	0.674	0.675	0.676	0.677	0.678	0.679	0.679	0.680	1	2	3	4	5	5	6	7	8
48	0.681	0.682	0.683	0.684	0.685	0.686	0.687	0.688	0.688	0.689	1	2	3	4	4	5	6	7	8
49	0.690	0.691	0.692	0.693	0.694	0.695	0.695	0.696	0.697	0.698	1	2	3	4	4	5	6	7	8
50	0.699	0.700	0.701	0.702	0.702	0.703	0.704	0.705	0.706	0.707	1	2	3	4	4	5	6	7	8
51	0.708	0.708	0.709	0.710	0.711	0.712	0.713	0.713	0.714	0.715	1	2	3	4	4	5	6	7	8
52	0.716	0.717	0.718	0.719	0.719	0.720	0.721	0.722	0.723	0.723	1	2	2	4	4	5	6	7	7
53	0.724	0.725	0.726	0.727	0.728	0.728	0.729	0.730	0.731	0.732	1	2	2	4	4	5	6	6	7
54	0.732	0.733	0.734	0.735	0.736	0.736	0.737	0.738	0.739	0.740	1	2	2	4	4	5	6	6	7

LOGARITHM TABLE

											Mean Difference								
55	0.740	0.741	0.742	0.743	0.744	0.744	0.745	0.746	0.747	0.747	1	2	2	4	4	5	5	6	7
56	0.748	0.749	0.750	0.751	0.751	0.752	0.753	0.754	0.754	0.755	1	2	2	4	4	5	5	6	7
57	0.756	0.757	0.757	0.758	0.759	0.760	0.760	0.761	0.762	0.763	1	2	2	4	4	5	5	6	7
58	0.763	0.764	0.765	0.766	0.766	0.767	0.768	0.769	0.769	0.770	1	1	2	4	4	4	5	6	7
59	0.771	0.772	0.772	0.773	0.774	0.775	0.775	0.776	0.777	0.777	1	1	2	3	4	4	5	6	7
60	0.778	0.779	0.780	0.780	0.781	0.782	0.782	0.783	0.784	0.785	1	1	2	3	4	4	5	6	6
61	0.785	0.786	0.787	0.787	0.788	0.789	0.790	0.790	0.791	0.792	1	1	2	3	4	4	5	6	6
62	0.792	0.793	0.794	0.794	0.795	0.796	0.797	0.797	0.798	0.799	1	1	2	3	3	4	5	6	6
63	0.799	0.800	0.801	0.801	0.802	0.803	0.803	0.804	0.805	0.806	1	1	2	3	3	4	5	5	6
64	0.806	0.807	0.808	0.808	0.809	0.810	0.810	0.811	0.812	0.812	1	1	2	3	3	4	5	5	6
65	0.813	0.814	0.814	0.815	0.816	0.816	0.817	0.818	0.818	0.819	1	1	2	3	3	4	5	5	6
66	0.820	0.820	0.821	0.822	0.822	0.823	0.823	0.824	0.825	0.825	1	1	2	3	3	4	5	5	6
67	0.826	0.827	0.827	0.828	0.829	0.829	0.830	0.831	0.831	0.832	1	1	2	3	3	4	5	5	6
68	0.833	0.833	0.834	0.834	0.835	0.836	0.836	0.837	0.838	0.838	1	1	2	3	3	4	4	5	6
69	0.839	0.839	0.840	0.841	0.841	0.842	0.843	0.843	0.844	0.844	1	1	2	2	3	4	4	5	6
70	0.845	0.846	0.846	0.847	0.848	0.848	0.849	0.849	0.850	0.851	1	1	2	2	3	4	4	5	6
71	0.851	0.852	0.852	0.853	0.854	0.854	0.855	0.856	0.856	0.857	1	1	2	2	3	4	4	5	5
72	0.857	0.858	0.859	0.859	0.860	0.860	0.861	0.862	0.862	0.863	1	1	2	2	3	4	4	5	5
73	0.863	0.864	0.865	0.865	0.866	0.866	0.867	0.867	0.868	0.869	1	1	2	2	3	4	4	5	5
74	0.869	0.870	0.870	0.871	0.872	0.872	0.873	0.873	0.874	0.874	1	1	2	2	3	4	4	5	5
75	0.875	0.876	0.876	0.877	0.877	0.878	0.879	0.879	0.880	0.880	1	1	2	2	3	3	4	5	5
76	0.881	0.881	0.882	0.883	0.883	0.884	0.884	0.885	0.885	0.886	1	1	2	2	3	3	4	5	5
77	0.886	0.887	0.888	0.888	0.889	0.889	0.890	0.890	0.891	0.892	1	1	2	2	3	3	4	4	5
78	0.892	0.893	0.893	0.894	0.894	0.895	0.895	0.896	0.897	0.897	1	1	2	2	3	3	4	4	5
79	0.898	0.898	0.899	0.899	0.900	0.900	0.901	0.901	0.902	0.903	1	1	2	2	3	3	4	4	5
80	0.903	0.904	0.904	0.905	0.905	0.906	0.906	0.907	0.907	0.908	1	1	2	2	3	3	4	4	5
81	0.908	0.909	0.910	0.910	0.911	0.911	0.912	0.912	0.913	0.913	1	1	2	2	3	3	4	4	5
82	0.914	0.914	0.915	0.915	0.916	0.916	0.917	0.918	0.918	0.919	1	1	2	2	3	3	4	4	5
83	0.919	0.920	0.920	0.921	0.921	0.922	0.922	0.923	0.923	0.924	1	1	2	2	3	3	4	4	5
84	0.924	0.925	0.925	0.926	0.926	0.927	0.927	0.928	0.928	0.929	1	1	2	2	3	3	4	4	5
85	0.929	0.930	0.930	0.931	0.931	0.932	0.932	0.933	0.933	0.934	1	1	2	2	3	3	4	4	5
86	0.934	0.935	0.936	0.936	0.937	0.937	0.938	0.938	0.939	0.939	1	1	2	2	3	3	4	4	5
87	0.940	0.940	0.941	0.941	0.942	0.942	0.943	0.943	0.943	0.944	0	1	1	2	2	3	3	4	5
88	0.944	0.945	0.945	0.946	0.946	0.947	0.947	0.948	0.948	0.949	0	1	1	2	2	3	3	4	4
89	0.949	0.950	0.950	0.951	0.951	0.952	0.952	0.953	0.953	0.954	0	1	1	2	2	3	3	4	4
90	0.954	0.955	0.955	0.956	0.956	0.957	0.957	0.958	0.958	0.959	0	1	1	2	2	3	3	4	4
91	0.959	0.960	0.960	0.960	0.961	0.961	0.962	0.962	0.963	0.963	0	1	1	2	2	3	3	4	4
92	0.964	0.964	0.965	0.965	0.966	0.966	0.967	0.967	0.968	0.968	0	1	1	2	2	3	3	4	4
93	0.968	0.969	0.969	0.970	0.970	0.971	0.971	0.972	0.972	0.973	0	1	1	2	2	3	3	4	4
94	0.973	0.974	0.974	0.975	0.975	0.975	0.976	0.976	0.977	0.977	0	1	1	2	2	3	3	4	4
95	0.978	0.978	0.979	0.979	0.980	0.980	0.980	0.981	0.981	0.982	0	1	1	2	2	3	3	4	4
96	0.982	0.983	0.983	0.984	0.984	0.985	0.985	0.985	0.986	0.986	0	1	1	2	2	3	3	4	4
97	0.987	0.987	0.988	0.988	0.989	0.989	0.989	0.990	0.990	0.991	0	1	1	2	2	3	3	4	4
98	0.991	0.992	0.992	0.993	0.993	0.993	0.994	0.994	0.995	0.995	0	1	1	2	2	3	3	4	4
99	0.996	0.996	0.997	0.997	0.997	0.998	0.998	0.999	0.999	1.000	0	1	1	2	2	3	3	3	4

ANTILOG TABLE

											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.00	1.000	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
0.01	1.023	1.026	1.028	1.030	1.033	1.035	1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
0.02	1.047	1.050	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
0.03	1.072	1.074	1.076	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
0.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
0.05	1.122	1.125	1.127	1.130	1.132	1.135	1.138	1.140	1.143	1.146	0	1	1	1	1	2	2	2	2
0.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
0.07	1.175	1.178	1.180	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
0.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
0.09	1.230	1.233	1.236	1.239	1.242	1.245	1.247	1.250	1.253	1.256	0	1	1	1	1	2	2	2	3
0.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
0.11	1.288	1.291	1.294	1.297	1.300	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
0.12	1.318	1.321	1.324	1.327	1.330	1.334	1.337	1.340	1.343	1.346	0	1	1	1	2	2	2	2	3
0.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
0.14	1.380	1.384	1.387	1.390	1.393	1.396	1.400	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
0.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
0.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
0.17	1.479	1.483	1.486	1.489	1.493	1.496	1.500	1.503	1.507	1.510	0	1	1	1	2	2	2	3	3
0.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
0.19	1.549	1.552	1.556	1.560	1.563	1.567	1.570	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
0.20	1.585	1.589	1.592	1.596	1.600	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
0.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
0.22	1.660	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.690	1.694	0	1	1	2	2	2	3	3	3
0.23	1.698	1.702	1.706	1.710	1.714	1.718	1.722	1.726	1.730	1.734	0	1	1	2	2	2	3	3	4
0.24	1.738	1.742	1.746	1.750	1.754	1.758	1.762	1.766	1.770	1.774	0	1	1	2	2	2	3	3	4
0.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
0.26	1.820	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
0.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
0.28	1.905	1.910	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
0.29	1.950	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
0.30	1.995	2.000	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
0.31	2.042	2.046	2.051	2.056	2.061	2.065	2.070	2.075	2.080	2.084	0	1	1	2	2	3	3	4	4
0.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
0.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
0.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
0.35	2.239	2.244	2.249	2.254	2.259	2.265	2.270	2.275	2.280	2.286	1	1	2	2	3	3	4	4	5
0.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	1	2	2	3	3	4	4	5
0.37	2.344	2.350	2.355	2.360	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
0.38	2.399	2.404	2.410	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
0.39	2.455	2.460	2.466	2.472	2.477	2.483	2.489	2.495	2.500	2.506	1	1	2	2	3	3	4	5	5
0.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
0.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
0.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6

ANTILOG TABLE

											Mean Difference								
0.50	3.162	3.170	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1	1	2	3	4	4	5	6	7
0.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
0.52	3.311	3.319	3.327	3.334	3.342	3.350	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
0.53	3.388	3.396	3.404	3.412	3.420	3.428	3.436	3.443	3.451	3.459	1	2	2	3	4	5	6	6	7
0.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.540	1	2	2	3	4	5	6	6	7
0.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
0.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.690	3.698	3.707	1	2	2	3	4	5	6	7	8
0.57	3.715	3.724	3.733	3.741	3.750	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
0.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
0.59	3.890	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
0.60	3.981	3.990	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
0.61	4.074	4.083	4.093	4.102	4.111	4.121	4.130	4.140	4.150	4.159	1	2	3	4	5	6	7	8	9
0.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
0.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
0.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.550	4.560	1	2	3	4	5	6	7	8	9
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
0.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	7	9	10
0.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	5	7	8	9	10
0.69	4.898	4.909	4.920	4.932	4.943	4.955	4.966	4.977	4.989	5.000	1	2	3	4	5	7	8	9	10
0.70	5.012	5.023	5.035	5.047	5.058	5.070	5.082	5.093	5.105	5.117	1	2	3	4	5	7	8	9	11
0.71	5.129	5.140	5.152	5.164	5.176	5.188	5.200	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
0.72	5.248	5.260	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
0.73	5.370	5.383	5.395	5.408	5.420	5.433	5.445	5.458	5.470	5.483	1	3	4	5	6	8	9	10	11
0.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.610	1	3	4	5	6	8	9	10	12
0.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
0.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
0.77	5.888	5.902	5.916	5.929	5.943	5.957	5.970	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
0.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
0.79	6.166	6.180	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.310	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.730	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
0.84	6.918	6.934	6.950	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
0.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2	3	5	7	8	10	12	13	15
0.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
0.87	7.413	7.430	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
0.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	3	5	7	9	10	12	14	16
0.89	7.762	7.780	7.798	7.816	7.834	7.852	7.870	7.889	7.907	7.925	2	4	5	7	9	11	12	14	16
0.90	7.943	7.962	7.980	7.998	8.017	8.035	8.054	8.072	8.091	8.110	2	4	6	7	9	11	13	15	17
0.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.260	8.279	8.299	2	4	6	8	9	11	13	15	17
0.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
0.93	8.511	8.531	8.551	8.570	8.590	8.610	8.630	8.650	8.670	8.690	2	4	6	8	10	12	14	16	18
0.94	8.710	8.730	8.750	8.770	8.790	8.810	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
0.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
0.96	9.120	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.290	9.311	2	4	6	8	11	13	15	17	19
0.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
0.98	9.550	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.750	2	4	7	9	11	13	16	18	20
0.99	9.772	9.795	9.817	9.840	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	20

NATURAL SINES

x	0	1	2	3	4	5	6	7	8	9
0	0	0.0017453	0.0034907	0.0052360	0.0069813	0.0087265	0.010472	0.012217	0.013962	0.015707
1	0.017452	0.019197	0.020942	0.022687	0.024432	0.026177	0.027922	0.029666	0.031411	0.033155
2	0.034899	0.036644	0.038388	0.040132	0.041876	0.043619	0.045363	0.047106	0.04885	0.050593
3	0.052336	0.054079	0.055822	0.057564	0.059306	0.061049	0.062791	0.064532	0.066274	0.068015
4	0.069756	0.071497	0.073238	0.074979	0.076719	0.078459	0.080199	0.081939	0.083678	0.085417
5	0.087156	0.088894	0.090633	0.092371	0.094108	0.095846	0.097583	0.09932	0.101056	0.102793
6	0.104528	0.106264	0.107999	0.109734	0.111469	0.113203	0.114937	0.116671	0.118404	0.120137
7	0.121869	0.123601	0.125333	0.127065	0.128796	0.130526	0.132256	0.133986	0.135716	0.137445
8	0.139173	0.140901	0.142629	0.144356	0.146083	0.147809	0.149535	0.151261	0.152986	0.15471
9	0.156434	0.158158	0.159881	0.161604	0.163326	0.165048	0.166769	0.168489	0.170209	0.171929
x	0	1	2	3	4	5	6	7	8	9
10	0.173648	0.175367	0.177085	0.178802	0.180519	0.182236	0.183951	0.185667	0.187381	0.189095
11	0.190809	0.192522	0.194234	0.195946	0.197657	0.199368	0.201078	0.202787	0.204496	0.206204
12	0.207912	0.209619	0.211325	0.21303	0.214735	0.21644	0.218143	0.219846	0.221548	0.22325
13	0.224951	0.226651	0.228351	0.23005	0.231748	0.233445	0.235142	0.236838	0.238533	0.240228
14	0.241922	0.243615	0.245307	0.246999	0.24869	0.25038	0.252069	0.253758	0.255446	0.257133
15	0.258819	0.260505	0.262189	0.263873	0.265556	0.267238	0.26892	0.2706	0.27228	0.273959
16	0.275637	0.277315	0.278991	0.280667	0.282341	0.284015	0.285688	0.287361	0.289032	0.290702
17	0.292372	0.29404	0.295708	0.297375	0.299041	0.300706	0.30237	0.304033	0.305695	0.307357
18	0.309017	0.310676	0.312335	0.313992	0.315649	0.317305	0.318959	0.320613	0.322266	0.323917
19	0.325568	0.327218	0.328867	0.330514	0.332161	0.333807	0.335452	0.337095	0.338738	0.34038
x	0	1	2	3	4	5	6	7	8	9
20	0.34202	0.34366	0.345298	0.346936	0.348572	0.350207	0.351842	0.353475	0.355107	0.356738
21	0.358368	0.359997	0.361625	0.363251	0.364877	0.366501	0.368125	0.369747	0.371368	0.372988
22	0.374607	0.376224	0.377841	0.379456	0.38107	0.382683	0.384295	0.385906	0.387516	0.389124
23	0.390731	0.392337	0.393942	0.395546	0.397148	0.398749	0.400349	0.401948	0.403545	0.405142
24	0.406737	0.40833	0.409923	0.411514	0.413104	0.414693	0.416281	0.417867	0.419452	0.421036
25	0.422618	0.424199	0.425779	0.427358	0.428935	0.430511	0.432086	0.433659	0.435231	0.436802
26	0.438371	0.439939	0.441506	0.443071	0.444635	0.446198	0.447759	0.449319	0.450878	0.452435
27	0.45399	0.455545	0.457098	0.45865	0.4602	0.461749	0.463296	0.464842	0.466387	0.46793
28	0.469472	0.471012	0.472551	0.474088	0.475624	0.477159	0.478692	0.480223	0.481754	0.483282
29	0.48481	0.486335	0.48786	0.489382	0.490904	0.492424	0.493942	0.495459	0.496974	0.498488
x	0	1	2	3	4	5	6	7	8	9
30	0.5	0.501511	0.50302	0.504528	0.506034	0.507538	0.509041	0.510543	0.512043	0.513541
31	0.515038	0.516533	0.518027	0.519519	0.52101	0.522499	0.523986	0.525472	0.526956	0.528438
32	0.529919	0.531399	0.532876	0.534352	0.535827	0.5373	0.538771	0.54024	0.541708	0.543174
33	0.544639	0.546102	0.547563	0.549023	0.550481	0.551937	0.553392	0.554844	0.556296	0.557745
34	0.559193	0.560639	0.562083	0.563526	0.564967	0.566406	0.567844	0.56928	0.570714	0.572146
35	0.573576	0.575005	0.576432	0.577858	0.579281	0.580703	0.582123	0.583541	0.584958	0.586372
36	0.587785	0.589196	0.590606	0.592013	0.593419	0.594823	0.596225	0.597625	0.599024	0.60042
37	0.601815	0.603208	0.604599	0.605988	0.607376	0.608761	0.610145	0.611527	0.612907	0.614285
38	0.615661	0.617036	0.618408	0.619779	0.621148	0.622515	0.62388	0.625243	0.626604	0.627963
39	0.62932	0.630676	0.632029	0.633381	0.634731	0.636078	0.637424	0.638768	0.64011	0.64145
x	0	1	2	3	4	5	6	7	8	9
40	0.642788	0.644124	0.645458	0.64679	0.64812	0.649448	0.650774	0.652098	0.653421	0.654741
41	0.656059	0.657375	0.658689	0.660002	0.661312	0.66262	0.663926	0.66523	0.666532	0.667833
42	0.669131	0.670427	0.671721	0.673013	0.674302	0.67559	0.676876	0.67816	0.679441	0.680721
43	0.681998	0.683274	0.684547	0.685818	0.687088	0.688355	0.68962	0.690882	0.692143	0.693402
44	0.694658	0.695913	0.697165	0.698415	0.699663	0.700909	0.702153	0.703395	0.704634	0.705872
45	0.707107	0.70834	0.709571	0.710799	0.712026	0.71325	0.714473	0.715693	0.716911	0.718126

NATURAL SINES

46	0.71934	0.720551	0.72176	0.722967	0.724172	0.725374	0.726575	0.727773	0.728969	0.730162
47	0.731354	0.732543	0.73373	0.734915	0.736097	0.737277	0.738455	0.739631	0.740805	0.741976
48	0.743145	0.744312	0.745476	0.746638	0.747798	0.748956	0.750111	0.751264	0.752415	0.753563
49	0.75471	0.755853	0.756995	0.758134	0.759271	0.760406	0.761538	0.762668	0.763796	0.764921
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
50	0.766044	0.767165	0.768284	0.7694	0.770513	0.771625	0.772734	0.77384	0.774944	0.776046
51	0.777146	0.778243	0.779338	0.78043	0.78152	0.782608	0.783693	0.784776	0.785857	0.786935
52	0.788011	0.789084	0.790155	0.791224	0.79229	0.793353	0.794415	0.795473	0.79653	0.797584
53	0.798636	0.799685	0.800731	0.801776	0.802817	0.803857	0.804894	0.805928	0.80696	0.80799
54	0.809017	0.810042	0.811064	0.812084	0.813101	0.814116	0.815128	0.816138	0.817145	0.81815
55	0.819152	0.820152	0.821149	0.822144	0.823136	0.824126	0.825113	0.826098	0.827081	0.82806
56	0.829038	0.830012	0.830984	0.831954	0.832921	0.833886	0.834848	0.835807	0.836764	0.837719
57	0.838671	0.83962	0.840567	0.841511	0.842452	0.843391	0.844328	0.845262	0.846193	0.847122
58	0.848048	0.848972	0.849893	0.850811	0.851727	0.85264	0.853551	0.854459	0.855364	0.856267
59	0.857167	0.858065	0.85896	0.859852	0.860742	0.861629	0.862514	0.863396	0.864275	0.865151
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
60	0.866025	0.866897	0.867765	0.868632	0.869495	0.870356	0.871214	0.872069	0.872922	0.873772
61	0.87462	0.875465	0.876307	0.877146	0.877983	0.878817	0.879649	0.880477	0.881303	0.882127
62	0.882948	0.883766	0.884581	0.885394	0.886204	0.887011	0.887815	0.888617	0.889416	0.890213
63	0.891007	0.891798	0.892586	0.893371	0.894154	0.894934	0.895712	0.896486	0.897258	0.898028
64	0.898794	0.899558	0.900319	0.901077	0.901833	0.902585	0.903335	0.904083	0.904827	0.905569
65	0.906308	0.907044	0.907777	0.908508	0.909236	0.909961	0.910684	0.911403	0.91212	0.912834
66	0.913545	0.914254	0.91496	0.915663	0.916363	0.91706	0.917755	0.918446	0.919135	0.919821
67	0.920505	0.921185	0.921863	0.922538	0.92321	0.92388	0.924546	0.92521	0.925871	0.926529
68	0.927184	0.927836	0.928486	0.929133	0.929776	0.930418	0.931056	0.931691	0.932324	0.932954
69	0.93358	0.934204	0.934826	0.935444	0.93606	0.936672	0.937282	0.937889	0.938493	0.939094
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
70	0.939693	0.940288	0.940881	0.941471	0.942057	0.942641	0.943223	0.943801	0.944376	0.944949
71	0.945519	0.946085	0.946649	0.94721	0.947768	0.948324	0.948876	0.949425	0.949972	0.950516
72	0.951057	0.951594	0.952129	0.952661	0.953191	0.953717	0.95424	0.954761	0.955278	0.955793
73	0.956305	0.956814	0.957319	0.957822	0.958323	0.95882	0.959314	0.959805	0.960294	0.960779
74	0.961262	0.961741	0.962218	0.962692	0.963163	0.96363	0.964095	0.964557	0.965016	0.965473
75	0.965926	0.966376	0.966823	0.967268	0.967709	0.968148	0.968583	0.969016	0.969445	0.969872
76	0.970296	0.970716	0.971134	0.971549	0.971961	0.97237	0.972776	0.973179	0.973579	0.973976
77	0.97437	0.974761	0.975149	0.975535	0.975917	0.976296	0.976672	0.977046	0.977416	0.977783
78	0.978148	0.978509	0.978867	0.979223	0.979575	0.979925	0.980271	0.980615	0.980955	0.981293
79	0.981627	0.981959	0.982287	0.982613	0.982935	0.983255	0.983571	0.983885	0.984196	0.984503
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
80	0.984808	0.985109	0.985408	0.985703	0.985996	0.986286	0.986572	0.986856	0.987136	0.987414
81	0.987688	0.98796	0.988228	0.988494	0.988756	0.989016	0.989272	0.989526	0.989776	0.990024
82	0.990268	0.990509	0.990748	0.990983	0.991216	0.991445	0.991671	0.991894	0.992115	0.992332
83	0.992546	0.992757	0.992966	0.993171	0.993373	0.993572	0.993768	0.993961	0.994151	0.994338
84	0.994522	0.994703	0.994881	0.995056	0.995227	0.995396	0.995562	0.995725	0.995884	0.996041
85	0.996195	0.996345	0.996493	0.996637	0.996779	0.996917	0.997053	0.997185	0.997314	0.997441
86	0.997564	0.997684	0.997801	0.997916	0.998027	0.998135	0.99824	0.998342	0.998441	0.998537
87	0.99863	0.998719	0.998806	0.99889	0.998971	0.999048	0.999123	0.999194	0.999263	0.999328
88	0.999391	0.99945	0.999507	0.99956	0.99961	0.999657	0.999701	0.999743	0.999781	0.999816
89	0.999848	0.999877	0.999903	0.999925	0.999945	0.999962	0.999976	0.999986	0.999994	0.999998
90	1	0.999998	0.999994	0.999986	0.999976	0.999962	0.999945	0.999925	0.999903	0.999877

NATURAL COSINES

x	0	1	2	3	4	5	6	7	8	9
0	1	0.999998	0.999994	0.999986	0.999976	0.999962	0.999945	0.999925	0.999903	0.999877
1	0.999848	0.999816	0.999781	0.999743	0.999701	0.999657	0.99961	0.99956	0.999507	0.99945
2	0.999391	0.999328	0.999263	0.999194	0.999123	0.999048	0.998971	0.99889	0.998806	0.998719
3	0.99863	0.998537	0.998441	0.998342	0.99824	0.998135	0.998027	0.997916	0.997801	0.997684
4	0.997564	0.997441	0.997314	0.997185	0.997053	0.996917	0.996779	0.996637	0.996493	0.996345
5	0.996195	0.996041	0.995884	0.995725	0.995562	0.995396	0.995227	0.995056	0.994881	0.994703
6	0.994522	0.994338	0.994151	0.993961	0.993768	0.993572	0.993373	0.993171	0.992966	0.992757
7	0.992546	0.992332	0.992115	0.991894	0.991671	0.991445	0.991216	0.990983	0.990748	0.990509
8	0.990268	0.990024	0.989776	0.989526	0.989272	0.989016	0.988756	0.988494	0.988228	0.98796
9	0.987688	0.987414	0.987136	0.986856	0.986572	0.986286	0.985996	0.985703	0.985408	0.985109
x	0	1	2	3	4	5	6	7	8	9
10	0.984808	0.984503	0.984196	0.983885	0.983571	0.983255	0.982935	0.982613	0.982287	0.981959
11	0.981627	0.981293	0.980955	0.980615	0.980271	0.979925	0.979575	0.979223	0.978867	0.978509
12	0.978148	0.977783	0.977416	0.977046	0.976672	0.976296	0.975917	0.975535	0.975149	0.974761
13	0.97437	0.973976	0.973579	0.973179	0.972776	0.97237	0.971961	0.971549	0.971134	0.970716
14	0.970296	0.969872	0.969445	0.969016	0.968583	0.968148	0.967709	0.967268	0.966823	0.966376
15	0.965926	0.965473	0.965016	0.964557	0.964095	0.96363	0.963163	0.962692	0.962218	0.961741
16	0.961262	0.960779	0.960294	0.959805	0.959314	0.95882	0.958323	0.957822	0.957319	0.956814
17	0.956305	0.955793	0.955278	0.954761	0.95424	0.953717	0.953191	0.952661	0.952129	0.951594
18	0.951057	0.950516	0.949972	0.949425	0.948876	0.948324	0.947768	0.94721	0.946649	0.946085
19	0.945519	0.944949	0.944376	0.943801	0.943223	0.942641	0.942057	0.941471	0.940881	0.940288
x	0	1	2	3	4	5	6	7	8	9
20	0.939693	0.939094	0.938493	0.937889	0.937282	0.936672	0.93606	0.935444	0.934826	0.934204
21	0.93358	0.932954	0.932324	0.931691	0.931056	0.930418	0.929776	0.929133	0.928486	0.927836
22	0.927184	0.926529	0.925871	0.92521	0.924546	0.92388	0.92321	0.922538	0.921863	0.921185
23	0.920505	0.919821	0.919135	0.918446	0.917755	0.91706	0.916363	0.915663	0.91496	0.914254
24	0.913545	0.912834	0.91212	0.911403	0.910684	0.909961	0.909236	0.908508	0.907777	0.907044
25	0.906308	0.905569	0.904827	0.904083	0.903335	0.902585	0.901833	0.901077	0.900319	0.899558
26	0.898794	0.898028	0.897258	0.896486	0.895712	0.894934	0.894154	0.893371	0.892586	0.891798
27	0.891007	0.890213	0.889416	0.888617	0.887815	0.887011	0.886204	0.885394	0.884581	0.883766
28	0.882948	0.882127	0.881303	0.880477	0.879649	0.878817	0.877983	0.877146	0.876307	0.875465
29	0.87462	0.873772	0.872922	0.872069	0.871214	0.870356	0.869495	0.868632	0.867765	0.866897
x	0	1	2	3	4	5	6	7	8	9
30	0.866025	0.865151	0.864275	0.863396	0.862514	0.861629	0.860742	0.859852	0.85896	0.858065
31	0.857167	0.856267	0.855364	0.854459	0.853551	0.85264	0.851727	0.850811	0.849893	0.848972
32	0.848048	0.847122	0.846193	0.845262	0.844328	0.843391	0.842452	0.841511	0.840567	0.83962
33	0.838671	0.837719	0.836764	0.835807	0.834848	0.833886	0.832921	0.831954	0.830984	0.830012
34	0.829038	0.82806	0.827081	0.826098	0.825113	0.824126	0.823136	0.822144	0.821149	0.820152
35	0.819152	0.81815	0.817145	0.816138	0.815128	0.814116	0.813101	0.812084	0.811064	0.810042
36	0.809017	0.80799	0.80696	0.805928	0.804894	0.803857	0.802817	0.801776	0.800731	0.799685
37	0.798636	0.797584	0.79653	0.795473	0.794415	0.793353	0.79229	0.791224	0.790155	0.789084
38	0.788011	0.786935	0.785857	0.784776	0.783693	0.782608	0.78152	0.78043	0.779338	0.778243
39	0.777146	0.776046	0.774944	0.77384	0.772734	0.771625	0.770513	0.7694	0.768284	0.767165
x	0	1	2	3	4	5	6	7	8	9
40	0.766044	0.764921	0.763796	0.762668	0.761538	0.760406	0.759271	0.758134	0.756995	0.755853
41	0.75471	0.753563	0.752415	0.751264	0.750111	0.748956	0.747798	0.746638	0.745476	0.744312
42	0.743145	0.741976	0.740805	0.739631	0.738455	0.737277	0.736097	0.734915	0.73373	0.732543
43	0.731354	0.730162	0.728969	0.727773	0.726575	0.725374	0.724172	0.722967	0.72176	0.720551
44	0.71934	0.718126	0.716911	0.715693	0.714473	0.71325	0.712026	0.710799	0.709571	0.70834
45	0.707107	0.705872	0.704634	0.703395	0.702153	0.700909	0.699663	0.698415	0.697165	0.695913

NATURAL COSINES

46	0.694658	0.693402	0.692143	0.690882	0.68962	0.688355	0.687088	0.685818	0.684547	0.683274
47	0.681998	0.680721	0.679441	0.67816	0.676876	0.67559	0.674302	0.673013	0.671721	0.670427
48	0.669131	0.667833	0.666532	0.66523	0.663926	0.66262	0.661312	0.660002	0.658689	0.657375
49	0.656059	0.654741	0.653421	0.652098	0.650774	0.649448	0.64812	0.64679	0.645458	0.644124
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
50	0.642788	0.64145	0.64011	0.638768	0.637424	0.636078	0.634731	0.633381	0.632029	0.630676
51	0.62932	0.627963	0.626604	0.625243	0.62388	0.622515	0.621148	0.619779	0.618408	0.617036
52	0.615661	0.614285	0.612907	0.611527	0.610145	0.608761	0.607376	0.605988	0.604599	0.603208
53	0.601815	0.60042	0.599024	0.597625	0.596225	0.594823	0.593419	0.592013	0.590606	0.589196
54	0.587785	0.586372	0.584958	0.583541	0.582123	0.580703	0.579281	0.577858	0.576432	0.575005
55	0.573576	0.572146	0.570714	0.56928	0.567844	0.566406	0.564967	0.563526	0.562083	0.560639
56	0.559193	0.557745	0.556296	0.554844	0.553392	0.551937	0.550481	0.549023	0.547563	0.546102
57	0.544639	0.543174	0.541708	0.54024	0.538771	0.5373	0.535827	0.534352	0.532876	0.531399
58	0.529919	0.528438	0.526956	0.525472	0.523986	0.522499	0.52101	0.519519	0.518027	0.516533
59	0.515038	0.513541	0.512043	0.510543	0.509041	0.507538	0.506034	0.504528	0.50302	0.501511
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
60	0.5	0.498488	0.496974	0.495459	0.493942	0.492424	0.490904	0.489382	0.48786	0.486335
61	0.48481	0.483282	0.481754	0.480223	0.478692	0.477159	0.475624	0.474088	0.472551	0.471012
62	0.469472	0.46793	0.466387	0.464842	0.463296	0.461749	0.4602	0.45865	0.457098	0.455545
63	0.45399	0.452435	0.450878	0.449319	0.447759	0.446198	0.444635	0.443071	0.441506	0.439939
64	0.438371	0.436802	0.435231	0.433659	0.432086	0.430511	0.428935	0.427358	0.425779	0.424199
65	0.422618	0.421036	0.419452	0.417867	0.416281	0.414693	0.413104	0.411514	0.409923	0.40833
66	0.406737	0.405142	0.403545	0.401948	0.400349	0.398749	0.397148	0.395546	0.393942	0.392337
67	0.390731	0.389124	0.387516	0.385906	0.384295	0.382683	0.38107	0.379456	0.377841	0.376224
68	0.374607	0.372988	0.371368	0.369747	0.368125	0.366501	0.364877	0.363251	0.361625	0.359997
69	0.358368	0.356738	0.355107	0.353475	0.351842	0.350207	0.348572	0.346936	0.345298	0.34366
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
70	0.34202	0.34038	0.338738	0.337095	0.335452	0.333807	0.332161	0.330514	0.328867	0.327218
71	0.325568	0.323917	0.322266	0.320613	0.318959	0.317305	0.315649	0.313992	0.312335	0.310676
72	0.309017	0.307357	0.305695	0.304033	0.30237	0.300706	0.299041	0.297375	0.295708	0.29404
73	0.292372	0.290702	0.289032	0.287361	0.285688	0.284015	0.282341	0.280667	0.278991	0.277315
74	0.275637	0.273959	0.27228	0.2706	0.26892	0.267238	0.265556	0.263873	0.262189	0.260505
75	0.258819	0.257133	0.255446	0.253758	0.252069	0.25038	0.24869	0.246999	0.245307	0.243615
76	0.241922	0.240228	0.238533	0.236838	0.235142	0.233445	0.231748	0.23005	0.228351	0.226651
77	0.224951	0.22325	0.221548	0.219846	0.218143	0.21644	0.214735	0.21303	0.211325	0.209619
78	0.207912	0.206204	0.204496	0.202787	0.201078	0.199368	0.197657	0.195946	0.194234	0.192522
79	0.190809	0.189095	0.187381	0.185667	0.183951	0.182236	0.180519	0.178802	0.177085	0.175367
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
80	0.173648	0.171929	0.170209	0.168489	0.166769	0.165048	0.163326	0.161604	0.159881	0.158158
81	0.156434	0.15471	0.152986	0.151261	0.149535	0.147809	0.146083	0.144356	0.142629	0.140901
82	0.139173	0.137445	0.135716	0.133986	0.132256	0.130526	0.128796	0.127065	0.125333	0.123601
83	0.121869	0.120137	0.118404	0.116671	0.114937	0.113203	0.111469	0.109734	0.107999	0.106264
84	0.104528	0.102793	0.101056	0.09932	0.097583	0.095846	0.094108	0.092371	0.090633	0.088894
85	0.087156	0.085417	0.083678	0.081939	0.080199	0.078459	0.076719	0.074979	0.073238	0.071497
86	0.069756	0.068015	0.066274	0.064532	0.062791	0.061049	0.059306	0.057564	0.055822	0.054079
87	0.052336	0.050593	0.04885	0.047106	0.045363	0.043619	0.041876	0.040132	0.038388	0.036644
88	0.034899	0.033155	0.031411	0.029666	0.027922	0.026177	0.024432	0.022687	0.020942	0.019197
89	0.017452	0.015707	0.013962	0.012217	0.010472	0.0087265	0.0069813	0.0052360	0.0034907	0.0017453
90	0	0.0017453	0.0034907	0.0052360	0.0069813	0.0087265	0.010472	0.012217	0.013962	0.015707

NATURAL TANGENTS

x	0	1	2	3	4	5	6	7	8	9
0	0	0.0017453	0.0034907	0.0052360	0.0069814	0.0087269	0.010472	0.012218	0.013964	0.015709
1	0.017455	0.019201	0.020947	0.022693	0.024439	0.026186	0.027933	0.029679	0.031426	0.033173
2	0.034921	0.036668	0.038416	0.040164	0.041912	0.043661	0.04541	0.047159	0.048908	0.050658
3	0.052408	0.054158	0.055909	0.05766	0.059411	0.061163	0.062915	0.064667	0.06642	0.068173
4	0.069927	0.071681	0.073435	0.07519	0.076946	0.078702	0.080458	0.082215	0.083972	0.08573
5	0.087489	0.089248	0.091007	0.092767	0.094528	0.096289	0.098051	0.099813	0.101576	0.10334
6	0.105104	0.106869	0.108635	0.110401	0.112168	0.113936	0.115704	0.117473	0.119243	0.121013
7	0.122785	0.124557	0.126329	0.128103	0.129877	0.131652	0.133428	0.135205	0.136983	0.138761
8	0.140541	0.142321	0.144102	0.145884	0.147667	0.149451	0.151236	0.153022	0.154808	0.156596
9	0.158384	0.160174	0.161965	0.163756	0.165549	0.167343	0.169137	0.170933	0.17273	0.174528
x	0	1	2	3	4	5	6	7	8	9
10	0.176327	0.178127	0.179928	0.181731	0.183534	0.185339	0.187145	0.188952	0.19076	0.19257
11	0.19438	0.196192	0.198005	0.19982	0.201635	0.203452	0.205271	0.20709	0.208911	0.210733
12	0.212557	0.214381	0.216208	0.218035	0.219864	0.221695	0.223526	0.22536	0.227194	0.229031
13	0.230868	0.232707	0.234548	0.23639	0.238234	0.240079	0.241925	0.243774	0.245624	0.247475
14	0.249328	0.251183	0.253039	0.254897	0.256756	0.258618	0.26048	0.262345	0.264211	0.266079
15	0.267949	0.269821	0.271694	0.273569	0.275446	0.277325	0.279205	0.281087	0.282971	0.284857
16	0.286745	0.288635	0.290527	0.29242	0.294316	0.296213	0.298113	0.300014	0.301918	0.303823
17	0.305731	0.30764	0.309552	0.311465	0.313381	0.315299	0.317219	0.319141	0.321065	0.322991
18	0.32492	0.32685	0.328783	0.330718	0.332656	0.334595	0.336537	0.338481	0.340428	0.342377
19	0.344328	0.346281	0.348237	0.350195	0.352156	0.354119	0.356084	0.358052	0.360022	0.361995
x	0	1	2	3	4	5	6	7	8	9
20	0.36397	0.365948	0.367928	0.369911	0.371897	0.373885	0.375875	0.377869	0.379864	0.381863
21	0.383864	0.385868	0.387874	0.389884	0.391896	0.39391	0.395928	0.397948	0.399971	0.401997
22	0.404026	0.406058	0.408092	0.41013	0.41217	0.414214	0.41626	0.418309	0.420361	0.422417
23	0.424475	0.426536	0.428601	0.430668	0.432739	0.434812	0.436889	0.438969	0.441053	0.443139
24	0.445229	0.447322	0.449418	0.451517	0.45362	0.455726	0.457836	0.459949	0.462065	0.464185
25	0.466308	0.468434	0.470564	0.472698	0.474835	0.476976	0.47912	0.481267	0.483419	0.485574
26	0.487733	0.489895	0.492061	0.494231	0.496404	0.498582	0.500763	0.502948	0.505136	0.507329
27	0.509525	0.511726	0.51393	0.516138	0.518351	0.520567	0.522787	0.525012	0.52724	0.529473
28	0.531709	0.53395	0.536195	0.538445	0.540698	0.542956	0.545218	0.547484	0.549755	0.55203
29	0.554309	0.556593	0.558881	0.561174	0.563471	0.565773	0.568079	0.57039	0.572705	0.575026
x	0	1	2	3	4	5	6	7	8	9
30	0.57735	0.57968	0.582014	0.584353	0.586697	0.589045	0.591398	0.593757	0.59612	0.598488
31	0.600861	0.603239	0.605622	0.60801	0.610403	0.612801	0.615204	0.617613	0.620026	0.622445
32	0.624869	0.627299	0.629734	0.632174	0.634619	0.63707	0.639527	0.641989	0.644456	0.646929
33	0.649408	0.651892	0.654382	0.656877	0.659379	0.661886	0.664398	0.666917	0.669442	0.671972
34	0.674509	0.677051	0.679599	0.682154	0.684714	0.687281	0.689854	0.692433	0.695018	0.69761
35	0.700208	0.702812	0.705422	0.708039	0.710663	0.713293	0.71593	0.718573	0.721223	0.723879
36	0.726543	0.729213	0.731889	0.734573	0.737264	0.739961	0.742666	0.745377	0.748096	0.750821
37	0.753554	0.756294	0.759041	0.761796	0.764558	0.767327	0.770104	0.772888	0.77568	0.778479
38	0.781286	0.7841	0.786922	0.789752	0.79259	0.795436	0.79829	0.801151	0.804021	0.806898
39	0.809784	0.812678	0.81558	0.818491	0.821409	0.824336	0.827272	0.830216	0.833169	0.83613
x	0	1	2	3	4	5	6	7	8	9
40	0.8391	0.842078	0.845066	0.848062	0.851067	0.854081	0.857104	0.860136	0.863177	0.866227
41	0.869287	0.872356	0.875434	0.878521	0.881619	0.884725	0.887842	0.890967	0.894103	0.897249
42	0.900404	0.903569	0.906745	0.90993	0.913125	0.916331	0.919547	0.922773	0.92601	0.929257
43	0.932515	0.935783	0.939063	0.942352	0.945653	0.948965	0.952287	0.955621	0.958966	0.962322
44	0.965689	0.969067	0.972458	0.975859	0.979272	0.982697	0.986134	0.989582	0.993043	0.996515
45	1	1.0035	1.00701	1.01053	1.01406	1.01761	1.02117	1.02474	1.02832	1.03192

NATURAL TANGENTS

46	1.03553	1.03915	1.04279	1.04644	1.0501	1.05378	1.05747	1.06117	1.06489	1.06862
47	1.07237	1.07613	1.0799	1.08369	1.08749	1.09131	1.09514	1.09899	1.10285	1.10672
48	1.11061	1.11452	1.11844	1.12238	1.12633	1.13029	1.13428	1.13828	1.14229	1.14632
49	1.15037	1.15443	1.15851	1.16261	1.16672	1.17085	1.175	1.17916	1.18334	1.18754
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
50	1.19175	1.19599	1.20024	1.20451	1.20879	1.2131	1.21742	1.22176	1.22612	1.2305
51	1.2349	1.23931	1.24375	1.2482	1.25268	1.25717	1.26169	1.26622	1.27077	1.27535
52	1.27994	1.28456	1.28919	1.29385	1.29853	1.30323	1.30795	1.31269	1.31745	1.32224
53	1.32704	1.33187	1.33673	1.3416	1.3465	1.35142	1.35637	1.36134	1.36633	1.37134
54	1.37638	1.38145	1.38653	1.39165	1.39679	1.40195	1.40714	1.41235	1.41759	1.42286
55	1.42815	1.43347	1.43881	1.44418	1.44958	1.45501	1.46046	1.46595	1.47146	1.47699
56	1.48256	1.48816	1.49378	1.49944	1.50512	1.51084	1.51658	1.52235	1.52816	1.534
57	1.53986	1.54576	1.5517	1.55766	1.56366	1.56969	1.57575	1.58184	1.58797	1.59414
58	1.60033	1.60657	1.61283	1.61914	1.62548	1.63185	1.63826	1.64471	1.6512	1.65772
59	1.66428	1.67088	1.67752	1.68419	1.69091	1.69766	1.70446	1.71129	1.71817	1.72509
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
60	1.73205	1.73905	1.7461	1.75319	1.76032	1.76749	1.77471	1.78198	1.78929	1.79665
61	1.80405	1.8115	1.81899	1.82654	1.83413	1.84177	1.84946	1.8572	1.86499	1.87283
62	1.88073	1.88867	1.89667	1.90472	1.91282	1.92098	1.9292	1.93746	1.94579	1.95417
63	1.96261	1.97111	1.97966	1.98828	1.99695	2.00569	2.01449	2.02335	2.03227	2.04125
64	2.0503	2.05942	2.0686	2.07785	2.08716	2.09654	2.106	2.11552	2.12511	2.13477
65	2.14451	2.15432	2.1642	2.17416	2.18419	2.1943	2.20449	2.21475	2.2251	2.23553
66	2.24604	2.25663	2.2673	2.27806	2.28891	2.29984	2.31086	2.32197	2.33317	2.34447
67	2.35585	2.36733	2.37891	2.39058	2.40235	2.41421	2.42618	2.43825	2.45043	2.4627
68	2.47509	2.48758	2.50018	2.51289	2.52571	2.53865	2.5517	2.56487	2.57815	2.59156
69	2.60509	2.61874	2.63252	2.64642	2.66046	2.67462	2.68892	2.70335	2.71792	2.73263
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
70	2.74748	2.76247	2.77761	2.79289	2.80833	2.82391	2.83965	2.85555	2.87161	2.88783
71	2.90421	2.92076	2.93748	2.95437	2.97144	2.98868	3.00611	3.02372	3.04152	3.0595
72	3.07768	3.09606	3.11464	3.13341	3.1524	3.17159	3.191	3.21063	3.23048	3.25055
73	3.27085	3.29139	3.31216	3.33317	3.35443	3.37594	3.39771	3.41973	3.44202	3.46458
74	3.48741	3.51053	3.53393	3.55761	3.5816	3.60588	3.63048	3.65538	3.68061	3.70616
75	3.73205	3.75828	3.78485	3.81177	3.83906	3.86671	3.89474	3.92316	3.95196	3.98117
76	4.01078	4.04081	4.07127	4.10216	4.1335	4.1653	4.19756	4.2303	4.26352	4.29724
77	4.33148	4.36623	4.40152	4.43735	4.47374	4.51071	4.54826	4.58641	4.62518	4.66458
78	4.70463	4.74534	4.78673	4.82882	4.87162	4.91516	4.95945	5.00451	5.05037	5.09704
79	5.14455	5.19293	5.24218	5.29235	5.34345	5.39552	5.44857	5.50264	5.55777	5.61397
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
80	5.67128	5.72974	5.78938	5.85024	5.91236	5.97576	6.04051	6.10664	6.17419	6.24321
81	6.31375	6.38587	6.45961	6.53503	6.61219	6.69116	6.77199	6.85475	6.93952	7.02637
82	7.11537	7.20661	7.30018	7.39616	7.49465	7.59575	7.69957	7.80622	7.91582	8.02848
83	8.14435	8.26355	8.38625	8.51259	8.64275	8.77689	8.9152	9.05789	9.20516	9.35724
84	9.51436	9.6768	9.84482	10.0187	10.1988	10.3854	10.5789	10.7797	10.9882	11.2048
85	11.4301	11.6645	11.9087	12.1632	12.4288	12.7062	12.9962	13.2996	13.6174	13.9507
86	14.3007	14.6685	15.0557	15.4638	15.8945	16.3499	16.8319	17.3432	17.8863	18.4645
87	19.0811	19.7403	20.4465	21.2049	22.0217	22.9038	23.8593	24.8978	26.0307	27.2715
88	28.6363	30.1446	31.8205	33.6935	35.8006	38.1885	40.9174	44.0661	47.7395	52.0807
89	57.29	63.6567	71.6151	81.847	95.4895	114.589	143.237	190.984	286.478	572.957
90	∞									

State Council of Educational Research and Training, Chennai-6
Higher Secondary First Year Physics volume 1
List of Subject Experts in Textbook Writing

Domain Expert & Mentor

Prof. Rita John
Professor and Head,
Department of Theoretical Physics
University of Madras, Chennai.

Reviewers

Dr. V. N. Mani.
Scientist E, Head (C-MET)
Department of Electronic &
Information Technology, Hyderabad,
Government of India.

Prof. P. Ravindran
Dept. of. Physics,
School of Basic and Applied Sciences,
Central University of Tamil Nadu,
Thiruvarur.

Dr. Rajeev Shesha Joshi
Assistant Professor
School of Physical Sciences,
Central University of Karnataka.

Dr. Joe Jesudurai
Associate Professor
Dean of Sciences,
Department of Physics,
Loyola College, Chennai-34.

Content Writers

Mr. C. Joseph Prabagar
Assistant Professor
PG & Research Department of Physics,
Loyola College, Chennai-34.

Dr. S. S.Naina Mohammed
Assistant Professor
PG & Research Department of Physics,
Government Arts College,
Udumalpet, Tiruppur Dt - 642 126,

Dr. R Sugaraj Samuel
Assistant Professor
PG & Research Department of Physics,
The New College, Royapettah,
Chennai-14.

Dr. P. Elangovan
Assistant Professor
PG & Research Department of Physics,
Pachaiyappa's college, Chennai

Mr. T. Thamaraiselvan
PGT, GBHSS, Aranthangi – 614 616
Pudukottai Dt

Content Readers

Dr. V. Sivamadhavi
Associate Professor
Bharathi Women's Collage (A)
Chennai

Mr. N. Thamaraikannan
PG Assistant, JGN Hr.Sec.School,
Tambaram, Kanchipuram Dt

Dr. S. Ravikasi Venkatraman
PG Assistant, Govt. Hr. Sec.School,
Semmancheri, Kanchipuram Dt

Mr. T. Subbiah
PG Assistant, Govt.Girls Hr.Sec. School,
Achirupakkam, Kanchipuram Dt.

Mr.M.Palanichamy
Head Master, Government Hr. Sec.
School
Krishnarayapuram Karur Dt.

Tamil Translators

Mr. M. Palanikumar,
PGT,MNMSP Senthil Kumara
Nadar HSS,
Mallankinaru – 626 109
Virudhunagar Dt.

Mr. A. Ganesh
PGT, Sundaram GHSS,
Thirumazhisai – 600 124
Thiruvallur Dt

Mr. S. Sritharan
PGT, GHSS, Annasalai
Rasipuram - 637408
Namakkal Dt.

Dr. K. Vasudevan
PGT, Govt.ADW.Hr.Sec.School,
Kalangani – 637 014, Namakkal Dt

Mr. E. Elangovan
PGT, J. G. GHSS,
Manali New Town – 600 103
Thiruvallur Dt.

ICT Coordinators

Mr.G.Bergin
PG Assistant (Physics),
GBHSS, Sayalgudi,
Ramanathapuram Dt.

Mr. D. Jeyaselvan
BT Assistant (Science),GHS, Varagur,
Thanjavur Dt.

ICT- Animation & Video Developer

Mr. Kennedy Bernajsha
Kennedy Bernajsha Lemurians Media
F2 Padmavaithi Flat, Thilaiganga Nagar
7th Street, Nanganallur, Chennai.

SCERT Subject Coordinators

Mrs. D. Shanmugasundari
Graduate Assistant
P.U.M.School, Melathulukkankulam
Kariyapatti Union, Virudhunagar Dt.

Mr. G. Arulraja
Graduate Assistant
MKV Govt.Boys.Hr.Sec.School
Arani,Thiruvallur Dt

EMIS Technology Team

R.M. Satheesh
State Coordinator Technical,
TN EMIS, Samagra Shiksha.

K.P. Sathya Narayana
IT Consultant,
TN EMIS , Samagra Shikaha

R. Arun Maruthi Selvan
Technical Project Consultant,
TN EMIS, Samagra Shiksha

Typist

Mrs.D.Kavitha

Art and Design Team

Layout
Udhaya info, Chennai

Illustration
K. Sasi Kumar

In-House
QC - Asker Ali

Co-ordination
Ramesh Munisamy

This book has been printed on 80 G.S.M.
Elegant Maplitho paper.
Printed by offset at: